# Mathematics II – Examples
## II. Differential Calculus of Several Variables

### II.4. Total differential and tangent plane

Notation: let us have a function $z = f(x, y)$. Then

the (total) differential of the function $f$ in the point $A = [x_0, y_0]$:

$$\mathrm{d}f(A) = \frac{\partial f}{\partial x}(A) \cdot (x - x_0) + \frac{\partial f}{\partial y}(A) \cdot (y - y_0)$$

Denote $\mathrm{d}x = x - x_0, \mathrm{d}y = y - y_0$. Then

$$\mathrm{d}f(A) = \frac{\partial f}{\partial x}(A) \cdot \mathrm{d}x + \frac{\partial f}{\partial y}(A) \cdot \mathrm{d}y$$

**Example 91**: Let $f(x, y) = \frac{y}{x} - \frac{x}{y}$.

a) Determine and sketch domains, where the function $f$ is differentiable.

b) Write the differential of $f$ in the point $A = [x_0, y_0]$.

**Example 92**: Determine total differential and an approximate increment of the function $z = \frac{y}{x}$ in in the point $A = [2, 1]$ for $\triangle x = 0.1$ and $\triangle y = 0.2$. Compare them.

**Example 93**: Using total differential compute approximate increment of the function $z = \operatorname{arctg} \frac{y}{x}$ in if $x$ changes from $x_0 = 1$ to $x_1 = 1.2$ and $y$ changes from $y_0 = -3$ to $y_1 = -3.1$.

**Example 94**: Approximate a value of the expression $\ln \left( \sqrt{9.03} - \sqrt{0.99} - 1 \right)$ using total differential of an appropriate function.

**Example 95**: Compute approximate value of the expression $0.98^{3.04}$ using total differential of an inappropriate function.

**Example 96**: Find an equation of both tangent plane $\tau$ and normal line $n$ to the graph of the function $z = 2x^2 - 4y^2$ in the point $T = [2, 1, ?]$. Compute approximate value of this function in the point $[2.2, 1.3]$.

**Example 97**: Find an equation of tangent plane $\tau$ to the surface $z = x^2 + xy - y^2 + x + 3$ and parallel to the given plane $\rho: \ 5x - 3y - z = 0$.

- **Compute approximate values of a given expressions using total differential:**

**Example 98**: $\sqrt[3]{7.95} \cdot \sqrt{8.96}$

**Example 99**: $\dfrac{\sqrt[4]{0.97}}{1.02^3 \cdot \sqrt[3]{0.99}}$

**Example 100**: $\sqrt{4.04} \cdot \ln 1.02 \cdot \operatorname{arctg} 0.9$

- **Find an equation of both tangent plane $\tau$ and normal line $n$ to the surface $z = f(x, y)$ in the point $T$:**

**Example 101**: $z = 4\sqrt{x^2 + y^2}$, $T = [3, 4, ?]$

**Example 102**: $z = xy$, $T = [0, 0, ?]$

**Example 103**: $z = x^2 \cdot \cos\frac{1}{y}$, $T = [1, \frac{2}{\pi}, ?]$

**Example 104**: $z = \frac{1}{x} \cdot \arcsin y$, $T = [\frac{1}{2}, \frac{\sqrt{2}}{2}, ?]$

- **Find an equation of tangent plane $\tau$ to the surface $z = f(x, y)$ and parallel to the plane $\rho$:**

**Example 105**: $z = 2x^2 - y^2$, $\rho:\ 8x - 6y - z - 15 = 0$

**Example 106**: $z = \ln(x^2 + 2y^2)$, $\rho:\ 2x - z + 5 = 0$

**Example 107**: $z = x^2 - y^2 + 6xy + 2x$, $\rho:\ 4x + 6y - z = 0$

## II.5. Derivatives and differentials of high order

**Example 108**: Find all partial derivatives of second order of the function $f(x, y) = xy^3 - y \cdot e^{x+y^2}$.

**Example 109**: Prove that the function $u = u(x, t) = \operatorname{arctg}(2x - t)$ satisfies the partial differential equation $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial t \partial x} = 0$ in $\mathbb{E}_2$.

**Example 110***: There is given the function $f(x, y)$

$$f(x,y) = \begin{cases} xy\frac{x^2-2y^2}{x^2+y^2} & \text{for } [x,y] \neq [0,0], \\ 0 & \text{for } [x,y] = [0,0]. \end{cases}$$

Show that $\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0)$.

**Example 111**: Let us consider the scalar field $\phi(x, y, z) = xy^2 + z^3 - xyz + 3$. Compute $\operatorname{grad}\phi$, $\operatorname{rot}\operatorname{grad}\phi$.

**Example 112**: Let us consider the vector field $\vec{f} = (U, V, W) = \left(xy, x^2 - z^2, \frac{y}{x+z}\right)$. Compute $\operatorname{div}\vec{f}$, $\operatorname{rot}\vec{f}$, $\operatorname{div}\operatorname{rot}\vec{f}$.

**Example 113***: Let the scalar field $\phi(x, y, z)$ has continuous partial derivatives of second order in a domain $D \subset \mathbb{E}_3$. Prove that $\operatorname{rot}\operatorname{grad}\phi = \vec{0}$ in $D$.

**Example 114***: Let the vector field $\vec{f} = (U, V, W)$ has continuous partial derivatives of second order in a domain $D \subset \mathbb{E}_3$. Prove that $\operatorname{div}\operatorname{rot}\vec{f} = 0$ in $D$.

- **Find differentials of the given order:**

**Example 115***: $z = \sin(2x + y), \quad \mathrm{d}^2 z = ?$

**Example 116***: $z = x^3 - y^3 - xy + y^2, \quad \mathrm{d}^3 z = ?$

**Example 117***: $u = \mathrm{e}^{2x-3y}, \quad \mathrm{d}^2 u(A) = ?, \quad \mathrm{d}^3 u(A) = ?, \quad \mathrm{d}^n u(A) = ?, \quad A = [0, 0].$

Differentials can be used in the iportant **Taylor theorem**:
*Let a function $f(x, y)$ is differentiable$(n + 1)$ times in any interior point of a rectangle $M$ with a center in the point $A = [x_0, y_0]$. Then for any point $[x, y] \in M$ there exists a point $[\xi, \eta] \in M$, such that*

$$f(x, y) = f(A) + d f(A) + \frac{d^2 f(A)}{2!} + \cdots + \frac{d^n f(A)}{n!} + R_{n+1},$$

*where* $d f(A) = d f(x_0, y_0) = \frac{\partial f}{\partial x}(A) \cdot (x - x_0) + \frac{\partial f}{\partial y}(A) \cdot (y - y_0),$

$$d^2 f(A) = \frac{\partial^2 f}{\partial x^2}(A) \cdot (x - x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(A) \cdot (x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(A) \cdot (y - y_0)^2,$$

$$\vdots$$

$$d^n f(A) = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^n f}{\partial x^k \partial y^{n-k}}(A) \cdot (x - x_0)^k (y - y_0)^{n-k},$$

$$R_{n+1} = \frac{1}{(n + 1)!} d^{n+1} f(\xi, \eta).$$

**Example 118***: Write Taylor expansion of the function $f(x, y) = x^3 - 3xy^2 + y^2 + 4x - 5y$ in a neighborhood of the point $A = [2, -1]$ and use the result for approximation of the value of the function $f$ in the point $[2.1, -1.1]$.

**Example 119***: Write Taylor expansion of the fourth order of the function $f(x, y) = \cos(x^2+y^2)$ in a neighborhood of the point $[0, 0]$.

- **Find partial derivatives of second order of the given function:**

**Example 120**: $\phi(s, t) = \ln(s^3 + t)$

**Example 121**: $\phi(x, t) = \frac{\cos x^2}{t}$

**Example 122**: $f(x, y) = \mathrm{e}^{ax+by}$

**Example 123**: Verify that the function $u(x, t) = \sin(x - ct)$ and the function $u(x, t) = \sin(\omega ct) \cdot \sin(\omega t)$ satisfy the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$

**Example 124**: Verify that the function $u(x, y) = \mathrm{e}^x \cdot \sin y$ satisfies the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

- **Expand the function $f(x,y)$ using Taylor's theorem in a neighborhood of a point $A$:**

**Example 125***: $f(x,y) = x^3 + 5x^2 - 6xy + 2y^2, \quad A = [1,-2]$

**Example 126***: $f(x,y) = x^2 + 3xy - y^3, \quad A = [2,-1]$

## II.6. Gradient. Directional derivative