Mathematics II – Examples

II.7.* Derivative of composed function

Let the functions $z = f(x, y),\ x = x(u, v),\ y = y(u, v)$ are differentiable. Then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \tag{1}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \tag{2}$$

Example 160: For the following differentiable functions $z = f(x, y),\ x = u\cos v,\ y = u\sin v$

a) Compute the derivative of composed function $\frac{\partial z}{\partial u},\ \frac{\partial z}{\partial v}$.
b) Calculate for the particular function $z = f(x, y) = \mathrm{e}^x \ln y$.

Example 161: $w = f(x^4 + y^4 - 2z^4)$. Compute $V = \frac{1}{4x^3}\frac{\partial w}{\partial x} + \frac{1}{4y^3}\frac{\partial w}{\partial y} + \frac{1}{4z^3}\frac{\partial w}{\partial z}$.

Example 162: Verify that the function $y = f(x + at) + g(x - at)$ satisfies the partial differential equation $\frac{\partial^2 y}{\partial t^2} - a^2\frac{\partial^2 y}{\partial x^2} = 0$, where a is a constant in $\mathbb{R}$. Assume that both f and g have continuous partial derivative of second order. [Hint: $u = x + at,\ v = x - at,\ y = f(u) + g(v)$]

Example 163: Verify that the function $z = f(\frac{y}{x})$ satisfies the equation $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$. Assume that f is a differentiable function.

Example 164: Let us have $z = f(u, v),\ u = x^2 - y^2,\ v = \mathrm{e}^{xy}$. Compute the following differential expression $W = y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$, where f is a differentiable function.

Example 165: Let $f(u, v) = x^y$, where $x = u^2 + v^2,\ y = uv + v^2$. Compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ in the point A, which has coordinations $u = 1,\ v = -1$.

II.9. Implicit functions

Example 166: Prove, that the equation $x^3 + y^3 = 2x^2 + xy - 1$ define unique implicit function $y = f(x)$ in a neighbourhood of the point $A = [1, 0]$. Compute $f'(1)$ and $f"(1)$.

Example 167: Write an equation of both a tangent line t and a normal line n in the point $A = [1, 1]$ to the curve, which is defined by implicit way by the equation $F(x, y) \equiv x^3y + y^3x + x^2y - 3 = 0$.