Brief outline:

A gentle intro into Markov processes

with examples

Brief outline:

- From the very basics ….

A gentle intro into Markov processes

with examples

Brief outline:

- From the very basics ….
- Markov property in discrete time: Markov chain

A gentle intro into Markov processes

with examples

Brief outline:

- From the very basics ….
- Markov property in discrete time: Markov chain
- Markov processes in continuous time

A gentle intro into Markov processes

with examples

Brief outline:

- From the very basics ….
- Markov property in discrete time: Markov chain
- Markov processes in continuous time
- Examples

A gentle intro into Markov processes

with examples

Brief outline:

- From the very basics ….
- Markov property in discrete time: Markov chain
- Markov processes in continuous time
- Examples

A gentle intro into Markov processes

with examples

Gejza Dohnal, CTU Prague, Centre of Advanced Aerospace Technologies

From the very basics ....

From the very basics ....

- Always we are getting about a *probability space* $(\Omega,\mathscr{F},P)$

From the very basics ....

- Always we are getting about a *probability space* $(\Omega,\mathscr{F},P)$

- Let us consider a *random variable* X: $(\Omega,\mathscr{F})$ –> $(R,\mathscr{B})$

… what is it?

From the very basics ....

- Always we are getting about a *probability space* $(\Omega,\mathscr{F},P)$
- Let us consider a *random variable* X: $(\Omega,\mathscr{F})$ –> $(R,\mathscr{B})$

  … what is it?
- Let us have a *probability measure* (law, distribution) P

  … what it means?

From the very basics ....

- Always we are getting about a *probability space* $(\Omega,\mathscr{F},P)$
- Let us consider a *random variable* X: $(\Omega,\mathscr{F}) \text{–>} (R,\mathscr{B})$

  … what is it?
- Let us have a *probability measure* (law, distribution) P

  … what it means?
- The probability measure determines a *conditional probability measure* $P(X\in A \mid X\in B) = P(X\in A\cap B)/P(X\in B)$ … how?

From the very basics ....

- Always we are getting about a *probability space* $(\Omega,\mathscr{F},P)$
- Let us consider a *random variable* X: $(\Omega,\mathscr{F}) \to (R,\mathscr{B})$
  … what is it?
- Let us have a *probability measure* (law, distribution) P
  … what it means?
- The probability measure determines a *conditional probability measure*
  $P(X\in A \mid X\in B) = P(X\in A\cap B)/P(X\in B)$ … how?
- The probability measure is characterised by some functions like cdf, pdf, hf, qf, etc. … why, when?

From the very basics ....

- Always we are getting about a *probability space* $(\Omega,\mathscr{F},P)$
- Let us consider a *random variable* X: $(\Omega,\mathscr{F})$ –> $(R,\mathscr{B})$
  … what is it?
- Let us have a *probability measure* (law, distribution) P
  … what it means?
- The probability measure determines a *conditional probability measure*
  $P(X\in A \mid X\in B) = P(X\in A\cap B)/P(X\in B)$ … how?
- The probability measure is characterised by some functions like cdf, pdf, hf, qf, etc. … why, when?
- We use several characteristics to describe probability law
  … which?

From the very basics ....

- Let us consider a sequence of random variables $\{X_i\}_{i=1,2,\ldots}$ or a random process $\{X_t\}_{t\geq 0}$

From the very basics ....

- Let us consider a sequence of random variables $\{X_i\}_{i=1,2,\ldots}$ or a random process $\{X_t\}_{t\geq 0}$
- … or a sequence $\{X_{ti}\}_{i=1,2,\ldots}$ of observations of the random in times $0\leq t_1 < t_2 < \ldots$ (denote $X_i = X_{ti}$, $i = 1, 2, \ldots$)

From the very basics ....

- Let us consider a sequence of random variables $\{X_i\}_{i=1,2,\ldots}$ or a random process $\{X_t\}_{t\geq 0}$
- … or a sequence $\{X_{ti}\}_{i=1,2,\ldots}$ of observations of the random in times $0\leq t_1 < t_2 < \ldots$ (denote $X_i = X_{ti}$, $i = 1, 2, \ldots$)
- We need more than one probability measure … it's too complicated … $P(X_1\leq x_1, X_2\leq x_2, \ldots, X_n\leq x_n) = F(x_1, x_2, \ldots, x_n)$ - n-dimensional distribution

From the very basics ....

- Let us consider a sequence of random variables $\{X_i\}_{i=1,2,\ldots}$ or a random process $\{X_t\}_{t\geq 0}$
- … or a sequence $\{X_{t_i}\}_{i=1,2,\ldots}$ of observations of the random in times $0\leq t_1 < t_2 < \ldots$ (denote $X_i = X_{t_i}$, $i = 1, 2, \ldots$)
- We need more than one probability measure … it's too complicated …
  $P(X_1\leq x_1,X_2\leq x_2,\ldots,X_n\leq x_n) = F(x_1,x_2,\ldots,x_n)$ - n-dimensional distribution
- There appears a question of independence …
  X is independent of Y $\Leftrightarrow P(X\leq x|Y\leq y) = P(X\leq x)\ \forall x,y$.

From the very basics ....

- Let us consider a sequence of random variables $\{X_i\}_{i=1,2,\ldots}$ or a random process $\{X_t\}_{t\geq 0}$
- … or a sequence $\{X_{ti}\}_{i=1,2,\ldots}$ of observations of the random in times $0 \leq t_1 < t_2 < \ldots$ (denote $X_i = X_{ti}$, $i = 1, 2, \ldots$)
- We need more than one probability measure … it's too complicated …
  $P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = F(x_1, x_2, \ldots, x_n)$ - n-dimensional distribution
- There appears a question of independence …
  X is independent of Y $\iff P(X \leq x | Y \leq y) = P(X \leq x)\ \forall x,y$.
- Dependence is characterised by some strange functions like acf, pacf, etc.

From the very basics ....

- Let us consider a sequence of random variables $\{X_i\}_{i=1,2,\ldots}$ or a random process $\{X_t\}_{t\geq 0}$
- … or a sequence $\{X_{ti}\}_{i=1,2,\ldots}$ of observations of the random in times $0\leq t_1 < t_2 < \ldots$ (denote $X_i = X_{ti}$, $i = 1, 2, \ldots$)
- We need more than one probability measure … it's too complicated …
  $P(X_1\leq x_1, X_2\leq x_2, \ldots, X_n\leq x_n) = F(x_1, x_2, \ldots, x_n)$ - n-dimensional distribution
- There appears a question of independence …
  X is independent of Y <=> $P(X\leq x|Y\leq y) = P(X\leq x)\ \forall x,y$.
- Dependence is characterised by some strange functions like acf, pacf, etc.
- What we can do with this? - Data model description, dependecy structure, forecasting, change detection, etc. etc.

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in some countable set $\mathcal{S}$

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in some countable set $\mathcal{S}$
- What is the probability of $[X_1 = a]$? … and of $[X_1 = a \wedge X_2 = b]$?

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in some countable set $\mathcal{S}$
- What is the probability of $[X_1 = a]$? … and of $[X_1 = a \wedge X_2 = b]$?
- … and of a sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ (trajectory)
  (i.e. the event $[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n]$) ?

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in some countable set $\mathcal{S}$
- What is the probability of $[X_1 = a]$? … and of $[X_1 = a \wedge X_2 = b]$?
- … and of a sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ (trajectory) (i.e. the event $[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n]$) ?
- in general case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

$$= P[X_1 = a_1].P[X_2 = a_2 | X_1 = a_1].P[X_3 = a_3 | X_2 = a_2 \wedge X_1 = a_1] \ldots$$

$$\ldots P[X_n = a_n \mid X_{n-1} = a_{n-1} \wedge X_{n-2} = a_{n-2} \wedge \ldots \wedge X_1 = a_1]$$

$$= p(a_1).p(a_2|a_1).p(a_3|a_2 a_1)\ldots p(a_n|a_{n-1}a_{n-2}\ldots a_2 a_1)$$

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in some countable set $\mathcal{S}$
- What is the probability of $[X_1 = a]$? … and of $[X_1 = a \wedge X_2 = b]$?
- … and of a sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ (trajectory) (i.e. the event $[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n]$) ?
- in general case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

$$= P[X_1 = a_1].P[X_2 = a_2|X_1 = a_1].P[X_3 = a_3|X_2 = a_2 \wedge X_1 = a_1] \ \ldots$$

$$\ldots \ P[X_n = a_n \mid X_{n-1} = a_{n-1} \wedge X_{n-2} = a_{n-2} \wedge \ldots \wedge X_1 = a_1]$$

$$= p(a_1).p(a_2|a_1).p(a_3|a_2a_1)\ldots p(a_n|a_{n-1}a_{n-2}\ldots a_2a_1)$$

- in i.i.d. case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

$$= P[X_1 = a_1].P[X_2 = a_2].P[X_3 = a_3]\ldots P[X_n = a_n]$$

$$= p(a_1).p(a_2).p(a_3)\ldots p(a_n)$$

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in some countable set $\mathcal{S}$
- What is the probability of $[X_1 = a]$? … and of $[X_1 = a \wedge X_2 = b]$?
- … and of a sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ (trajectory) (i.e. the event $[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n]$) ?
- in general case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

$$= P[X_1 = a_1].P[X_2 = a_2|X_1 = a_1].P[X_3 = a_3|X_2 = a_2 \wedge X_1 = a_1] \ldots$$

$$\ldots P[X_n = a_n \mid X_{n-1} = a_{n-1} \wedge X_{n-2} = a_{n-2} \wedge \ldots \wedge X_1 = a_1]$$

$$= p(a_1).p(a_2|a_1).p(a_3|a_2a_1)\ldots p(a_n|a_{n-1}a_{n-2}\ldots a_2a_1)$$

- in i.i.d. case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

$$= P[X_1 = a_1].P[X_2 = a_2].P[X_3 = a_3]\ldots P[X_n = a_n]$$

$$= p(a_1).p(a_2).p(a_3)\ldots p(a_n)$$

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in some countable set $\mathcal{S}$
- What is the probability of $[X_1 = a]$? … and of $[X_1 = a \wedge X_2 = b]$?
- … and of a sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ (trajectory) (i.e. the event $[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n]$) ?
- in general case: $\mathrm{P}[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$ ☹

$$= \mathrm{P}[X_1 = a_1].\mathrm{P}[X_2 = a_2 | X_1 = a_1].\mathrm{P}[X_3 = a_3 | X_2 = a_2 \wedge X_1 = a_1] \ \ldots$$

$$\ldots \ \mathrm{P}[X_n = a_n \mid X_{n-1} = a_{n-1} \wedge X_{n-2} = a_{n-2} \wedge \ldots \wedge X_1 = a_1]$$

$$= \mathrm{p}(a_1).\mathrm{p}(a_2|a_1).\mathrm{p}(a_3|a_2a_1)\ldots\mathrm{p}(a_n|a_{n-1}a_{n-2}\ldots a_2a_1)$$

- in i.i.d. case: $\mathrm{P}[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

$$= \mathrm{P}[X_1 = a_1].\mathrm{P}[X_2 = a_2].\mathrm{P}[X_3 = a_3]\ldots\mathrm{P}[X_n = a_n]$$

$$= \mathrm{p}(a_1).\mathrm{p}(a_2).\mathrm{p}(a_3)\ldots\mathrm{p}(a_n)$$

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in some countable set $\mathcal{S}$
- What is the probability of $[X_1 = a]$? … and of $[X_1 = a \wedge X_2 = b]$?
- … and of a sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ (trajectory) (i.e. the event $[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n]$) ?
- in general case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$ ☹

$$= P[X_1 = a_1].P[X_2 = a_2 | X_1 = a_1].P[X_3 = a_3 | X_2 = a_2 \wedge X_1 = a_1] \ \ldots$$

$$\ldots P[X_n = a_n \mid X_{n-1} = a_{n-1} \wedge X_{n-2} = a_{n-2} \wedge \ldots \wedge X_1 = a_1]$$

$$= p(a_1).p(a_2|a_1).p(a_3|a_2 a_1)\ldots p(a_n|a_{n-1}a_{n-2}\ldots a_2 a_1)$$

- in i.i.d. case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

☺

$$= P[X_1 = a_1].P[X_2 = a_2].P[X_3 = a_3]\ldots P[X_n = a_n]$$

$$= p(a_1).p(a_2).p(a_3)\ldots p(a_n)$$

A random sequence of discrete variables: the basic question

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in some countable set $\mathcal{S}$
- What is the probability of $[X_1 = a]$? … and of $[X_1 = a \wedge X_2 = b]$?
- … and of a sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ (trajectory)
(i.e. the event $[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n]$) ?
- in general case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

$= P[X_1 = a_1].P[X_2 = a_2|X_1 = a_1].P[X_3 = a_3|X_2 = a_2 \wedge X_1 = a_1] \ldots$

$\ldots P[X_n = a_n \mid X_{n-1} = a_{n-1} \wedge X_{n-2} = a_{n-2} \wedge \ldots \wedge X_1 = a_1]$

$= p(a_1).p(a_2|a_1).p(a_3|a_2a_1)\ldots p(a_n|a_{n-1}a_{n-2}\ldots a_2a_1)$

- in i.i.d. case: $P[X_1 = a_1 \wedge X_2 = a_2 \wedge \ldots \wedge X_n = a_n] =$

$= P[X_1 = a_1].P[X_2 = a_2].P[X_3 = a_3]\ldots P[X_n = a_n]$

$= p(a_1).p(a_2).p(a_3)\ldots p(a_n)$

Markov property in discrete time

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in $\mathcal{S}$

Andrej Andrejevič Markov
1856 - 1922

Markov property in discrete time

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\ldots}$ with values in $\mathcal{S}$
- We consider a simplified dependency structure: „the future state of the process depends only on the present state and it is independent of previous trajectory"

Andrej Andrejevič Markov
1856 - 1922

Markov property in discrete time

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in S
- We consider a simplified dependency structure: „the future state of the process depends only on the present state and it is independent of previous trajectory"
- When we are in the time t_n, the Markov property says:

$$\boxed{P[X_{n+1}=a_{n+1} | X_n=a_n \wedge X_{n-1}=a_{n-1} \wedge \dots \wedge X_0=a_0] = P[X_{n+1}=a_{n+1} \mid X_n=a_n]}$$

Andrej Andrejevič Markov
1856 - 1922

Markov property in discrete time

Andrej Andrejevič Markov
1856 - 1922

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in S
- We consider a simplified dependency structure: „the future state of the process depends only on the present state and it is independent of previous trajectory"
- When we are in the time t_n, the Markov property says:

$$\boxed{P[X_{n+1}=a_{n+1}|X_n=a_n \wedge X_{n-1}=a_{n-1} \wedge \dots \wedge X_0=a_0] = P[X_{n+1}=a_{n+1} \mid X_n=a_n]}$$

- Then we have: $P[X_0 = a_0 \wedge X_1 = a_1 \wedge \dots \wedge X_n = a_n] =$

$$= P[X_0 = a_0].P[X_1 = a_1|X_0 = a_0].P[X_2 = a_2|X_1 = a_1]\dots P[X_n = a_n \mid X_{n-1} = a_{n-1}]$$

$$= p(a_0).p(a_1|a_0).p(a_2|a_1)\dots p(a_n|a_{n-1})$$

Markov property in discrete time

Andrej Andrejevič Markov
1856 - 1922

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in S
- We consider a simplified dependency structure: „the future state of the process depends only on the present state and it is independent of previous trajectory“
- When we are in the time t_n, the Markov property says:

$$\boxed{P[X_{n+1}=a_{n+1}|X_n=a_n \land X_{n-1}=a_{n-1} \land \dots \land X_0=a_0] = P[X_{n+1}=a_{n+1} \mid X_n=a_n]}$$

- Then we have: $P[X_0 = a_0 \land X_1 = a_1 \land \dots \land X_n = a_n] =$

$$= P[X_0 = a_0].P[X_1 = a_1|X_0 = a_0].P[X_2 = a_2|X_1 = a_1]\dots P[X_n = a_n \mid X_{n-1} = a_{n-1}]$$

$$= p(a_0).p(a_1|a_0).p(a_2|a_1)\dots p(a_n|a_{n-1})$$

Markov property in discrete time

Andrej Andrejevič Markov
1856 - 1922

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in $\mathcal{S}$
- We consider a simplified dependency structure: „the future state of the process depends only on the present state and it is independent of previous trajectory“
- When we are in the time t_n, the Markov property says:

$$\boxed{P[X_{n+1}=a_{n+1}|X_n=a_n \wedge X_{n-1}=a_{n-1} \wedge \dots \wedge X_0=a_0] = P[X_{n+1}=a_{n+1} \mid X_n=a_n]}$$

- Then we have: $P[X_0 = a_0 \wedge X_1 = a_1 \wedge \dots \wedge X_n = a_n] =$

$$= P[X_0 = a_0].P[X_1 = a_1|X_0 = a_0].P[X_2 = a_2|X_1 = a_1]\dots P[X_n = a_n \mid X_{n-1} = a_{n-1}]$$

$$= p(a_0).p(a_1|a_0).p(a_2|a_1)\dots p(a_n|a_{n-1})$$

initial probability

Markov property in discrete time

Andrej Andrejevič Markov
1856 - 1922

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in $\mathcal{S}$
- We consider a simplified dependency structure: „the future state of the process depends only on the present state and it is independent of previous trajectory“
- When we are in the time t_n, the Markov property says:

$$\boxed{P[X_{n+1}=a_{n+1}|X_n=a_n \wedge X_{n-1}=a_{n-1} \wedge \dots \wedge X_0=a_0] = P[X_{n+1}=a_{n+1} \mid X_n=a_n]}$$

- Then we have: $P[X_0 = a_0 \wedge X_1 = a_1 \wedge \dots \wedge X_n = a_n] =$

$$= P[X_0 = a_0].P[X_1 = a_1|X_0 = a_0].P[X_2 = a_2|X_1 = a_1]\dots P[X_n = a_n \mid X_{n-1} = a_{n-1}]$$

$$= p(a_0).p(a_1|a_0).p(a_2|a_1)\dots p(a_n|a_{n-1})$$

initial probability

transition probabilities

Markov property in discrete time

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in $\mathcal{S}$
- We consider a simplified dependency structure: „the future state of the process depends only on the present state and it is independent of previous trajectory“
- When we are in the time t_n, the Markov property says:

$$\boxed{P[X_{n+1}=a_{n+1}|X_n=a_n \wedge X_{n-1}=a_{n-1} \wedge \dots \wedge X_0=a_0] = P[X_n=a_n \mid X_{n-1}=a_{n-1}]}$$

- Then we have: $P[X_0 = a_0 \wedge X_1 = a_1 \wedge \dots \wedge X_n = a_n] =$ $= P[X_0 = a_0].P[X_1 = a_1|X_0 = a_0].P[X_2 = a_2|X_1 = a_1]\dots P[X_n = a_n \mid X_{n-1} = a_{n-1}]$
- Denote the *initial distribution*, $\boldsymbol{p} = (p_1, \dots, p_n)$, where $p_i = P[X_0 = a_i]$

Markov property in discrete time

- Let us consider a sequence of discrete random variables $\{X_i\}_{i=1,2,\dots}$ with values in $\mathcal{S}$
- We consider a simplified dependency structure:
  „the future state of the process depends only on the present state and it is independent of previous trajectory“
- When we are in the time t_n, the Markov property says:

$$\boxed{P[X_{n+1}=a_{n+1}|X_n=a_n \wedge X_{n-1}=a_{n-1} \wedge \dots \wedge X_0=a_0] = P[X_n=a_n \mid X_{n-1}=a_{n-1}]}$$

- Then we have: $P[X_0 = a_0 \wedge X_1 = a_1 \wedge \dots \wedge X_n = a_n] =$
  $= P[X_0 = a_0].P[X_1 = a_1|X_0 = a_0].P[X_2 = a_2|X_1 = a_1]\dots P[X_n = a_n \mid X_{n-1} = a_{n-1}]$
- Denote the *initial distribution*, $\boldsymbol{p}=(p_1, \dots, p_n)$, where $p_i = P[X_0 = a_i]$
- Denote the *transition probability matrix* $\mathbf{P}(k) = \{p_{ij}(k)\}_{i,j=1,2,\dots,n}$
  where $p_{ij}(k) = P[X_k = a_j|X_{k-1} = a_i]$

Markov process in discrete time - Markov chain (homogeneous case)

- Let us consider a finite set of states: $\mathcal{S} = \{a_1, a_2, ..., a_n\}$ along with the initial distribution, $\boldsymbol{p} = (p_1, \ldots, p_n)$, where $p_i = \mathrm{P}[X_0 = a_i]$

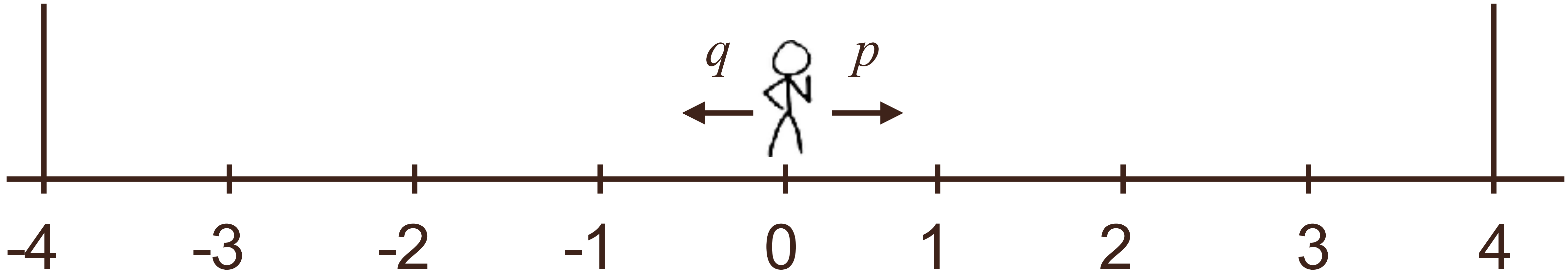
Markov process in discrete time - Markov chain (homogeneous case)

- Let us consider a finite set of states: $\mathcal{S} = \{a_1, a_2, \ldots, a_n\}$ along with the initial distribution, $\boldsymbol{p} = (p_1, \ldots, p_n)$, where $p_i = \mathrm{P}[X_0 = a_i]$
- In homogeneous case $p_{ij}(k) = p_{ij}$, i.e. it does not depend on the time. Then the transition probability matrices $\mathbf{P}(k)$ are the same for all $k = 1, 2, \ldots$, and we talk about the homogeneous Markov chain with the transition probability matrix $\mathbf{P}$.

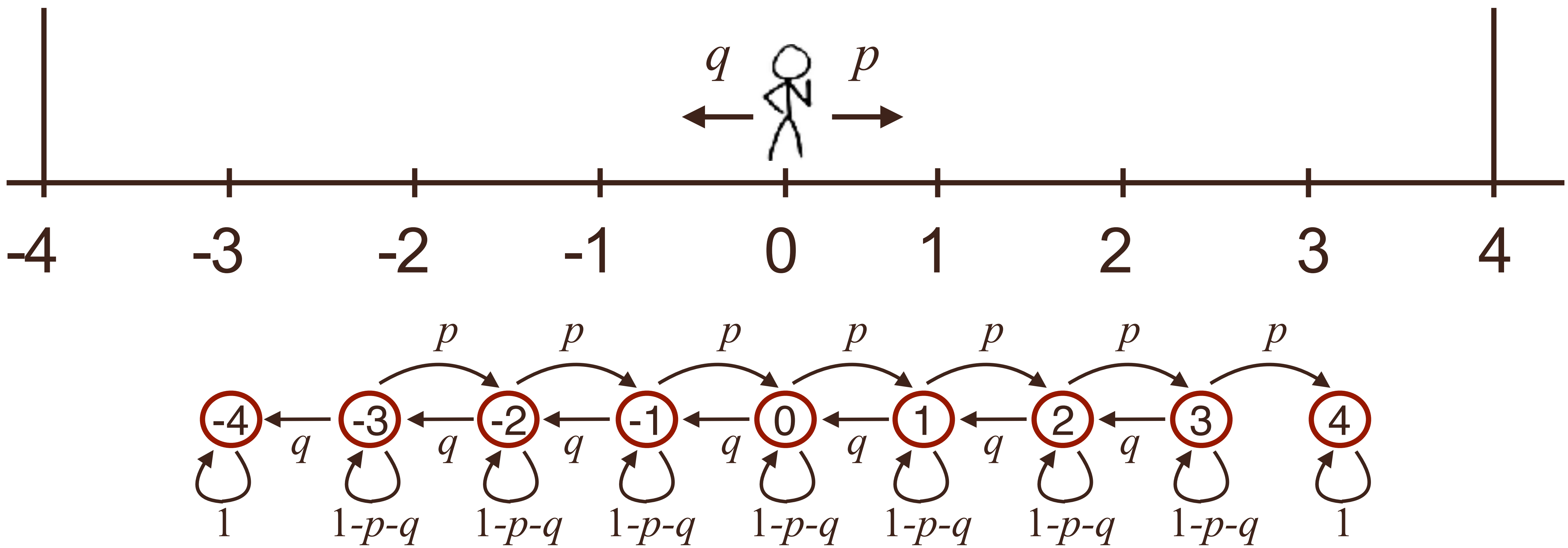
Markov process in discrete time - Markov chain (homogeneous case)

- Let us consider a finite set of states: $\mathcal{S} = \{a_1, a_2, \ldots, a_n\}$ along with the initial distribution, $\boldsymbol{p} = (p_1, \ldots, p_n)$, where $p_i = \mathrm{P}[X_0 = a_i]$
- In homogeneous case $p_{ij}(k) = p_{ij}$, i.e. it does not depend on the time. Then the transition probability matrices $\mathbf{P}(k)$ are the same for all $k = 1, 2, \ldots$, and we talk about the homogeneous Markov chain with the transition probability matrix $\mathbf{P}$.
- The Markov chain is determined by the triple ($\mathcal{S}$, $\boldsymbol{p}$, $\mathbf{P}$).

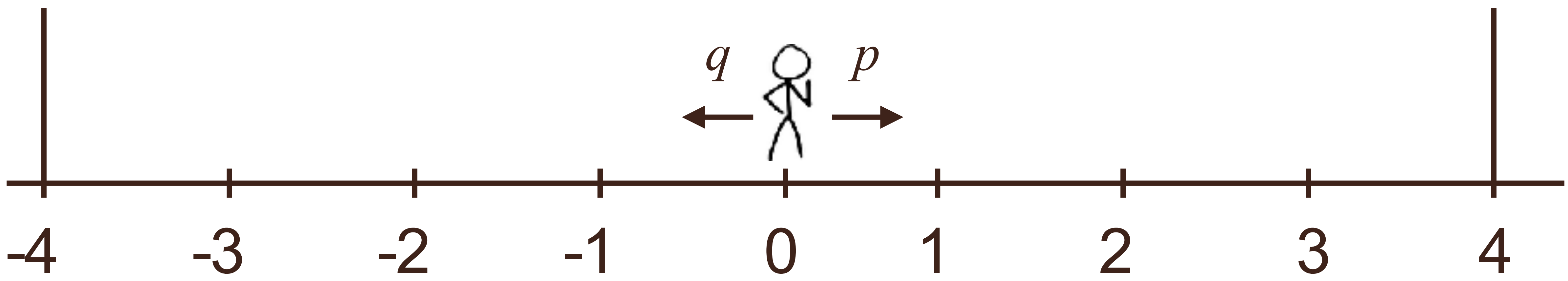
Markov chain - the random walk with absorbent walls

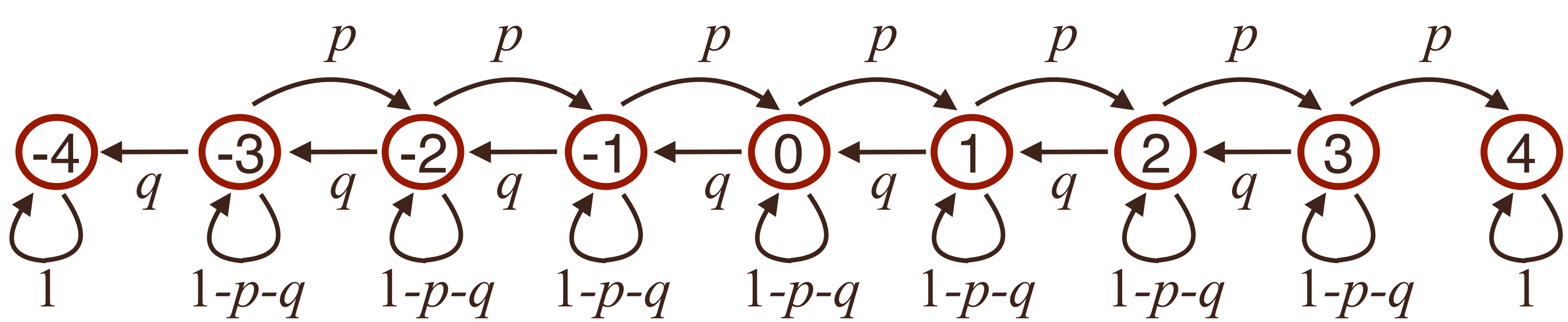


Markov chain - the random walk with absorbent walls



Markov chain - the random walk with absorbent walls

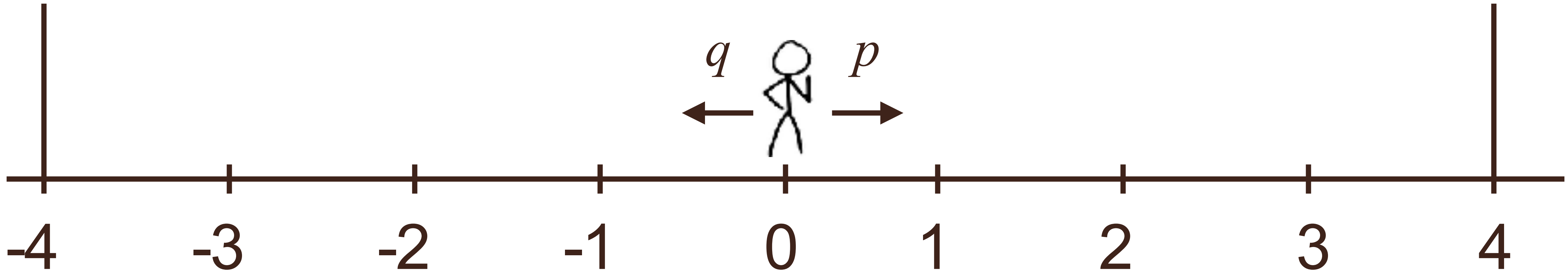


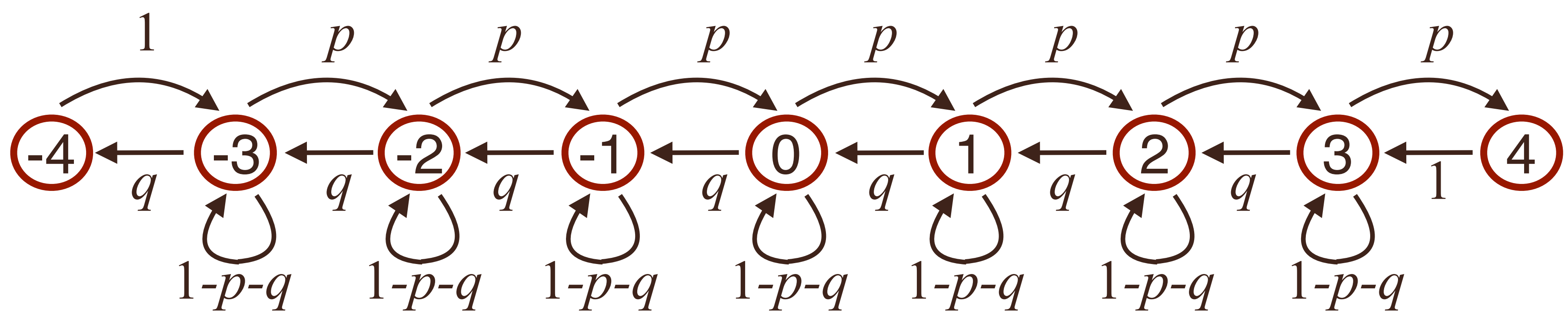


$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 1-p-q & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 1-p-q & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 1-p-q & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 1-p-q & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 1-p-q & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 1-p-q & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 1-p-q & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{p} = (0, 0, 0, 0, 1, 0, 0, 0, 0)$$

Markov chain - the random walk with reflecting walls

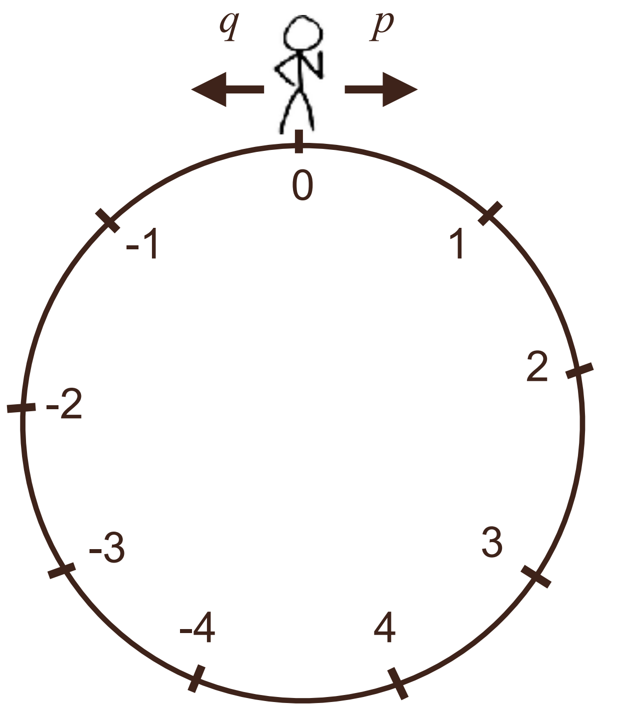




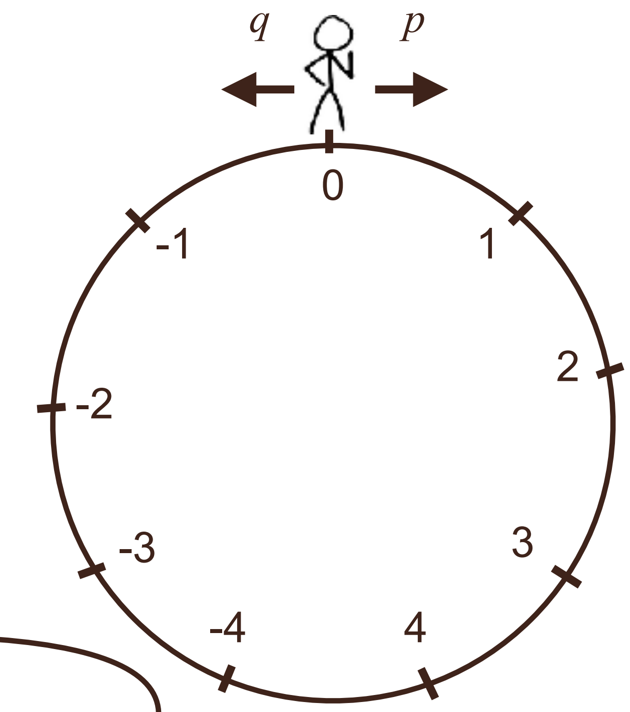
$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 1-p-q & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 1-p-q & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 1-p-q & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 1-p-q & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 1-p-q & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 1-p-q & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 1-p-q & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

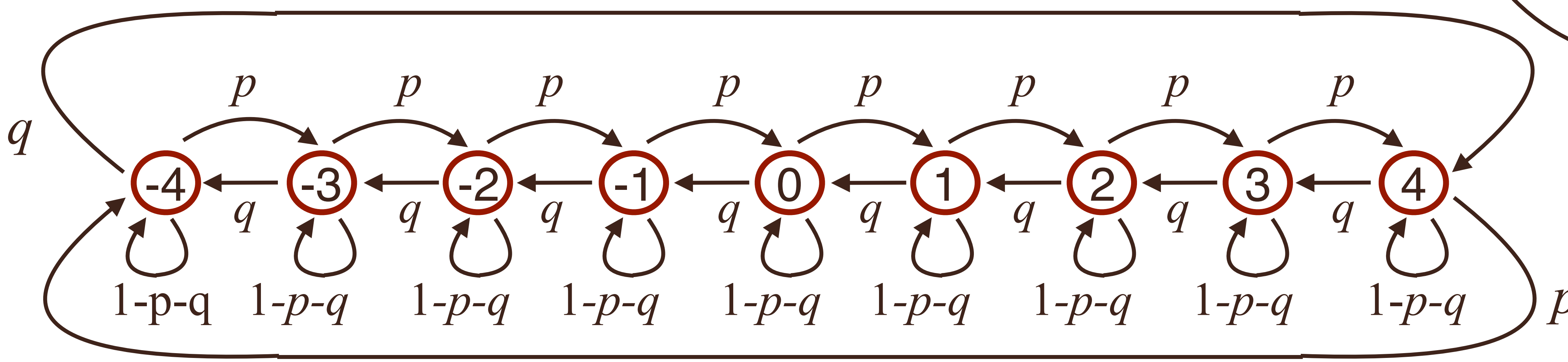
$$\boldsymbol{p} = (0, 0, 0, 0, 1, 0, 0, 0, 0)$$

Markov chain - the random walk on a circle

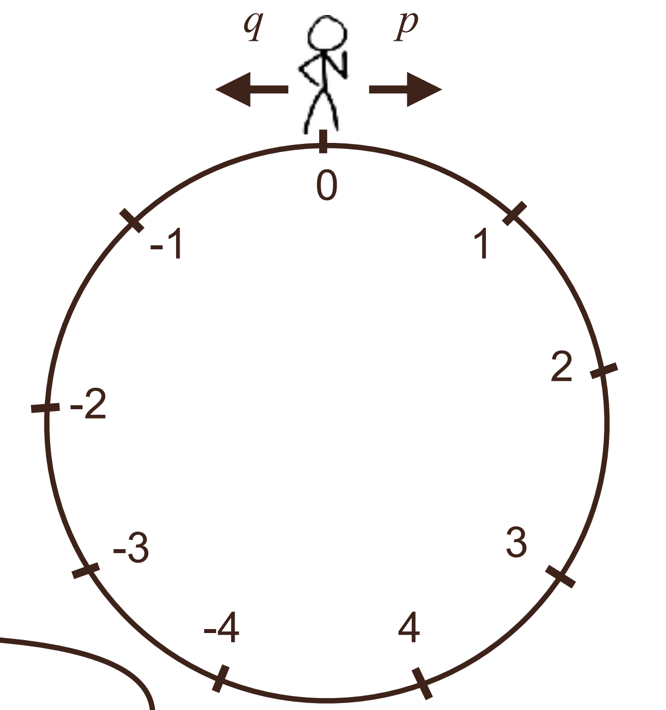


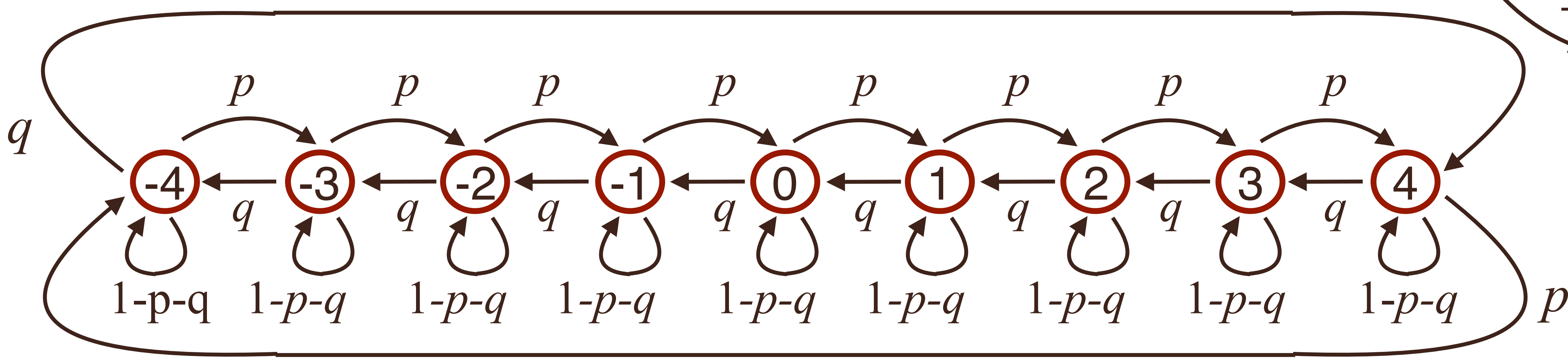
Markov chain - the random walk on a circle





Markov chain - the random walk on a circle

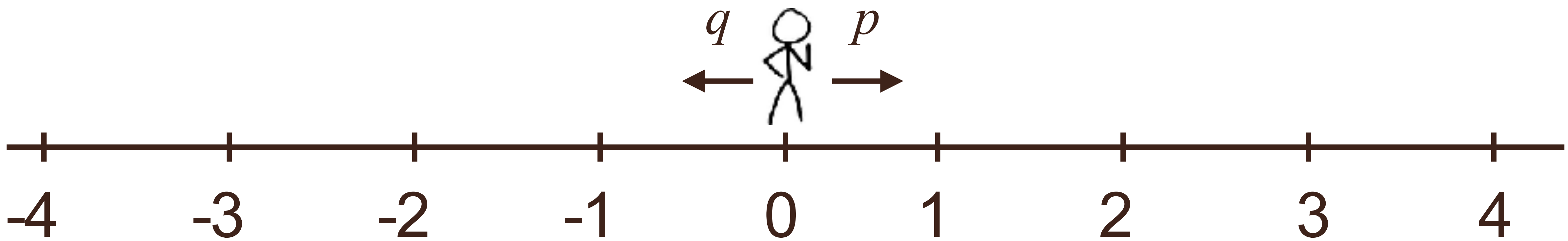


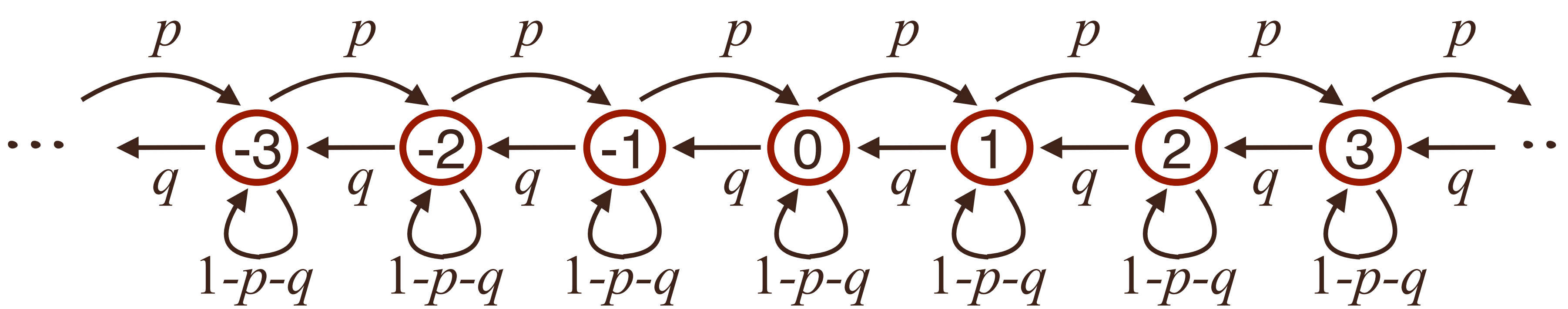


$$
\mathbf{P} = \begin{pmatrix}
1-p-q & p & 0 & 0 & 0 & 0 & 0 & 0 & q \\
q & 1-p-q & p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q & 1-p-q & p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q & 1-p-q & p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q & 1-p-q & p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q & 1-p-q & p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q & 1-p-q & p & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q & 1-p-q & p \\
p & 0 & 0 & 0 & 0 & 0 & 0 & q & 1-p-q
\end{pmatrix}
$$

$$
\boldsymbol{p} = (0, 0, 0, 0, 1, 0, 0, 0, 0)
$$

Markov chain - the random walk without bounds



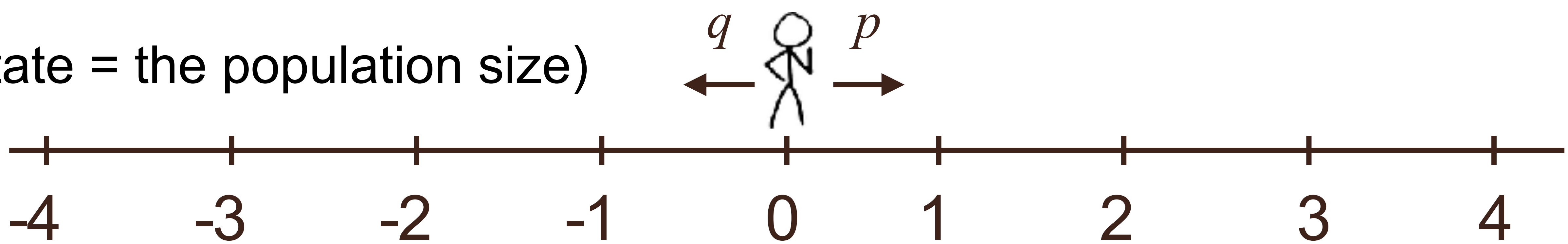


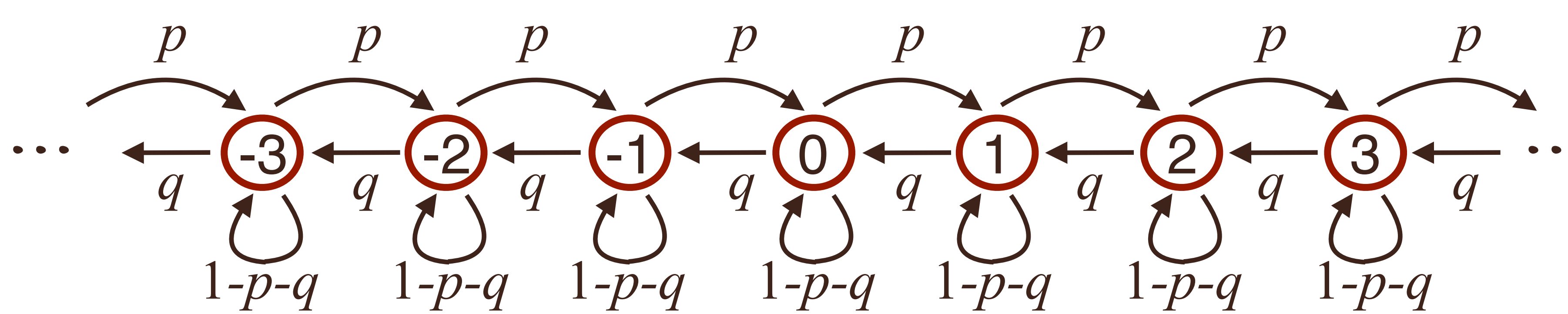
$$\mathbf{P} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1-p-q & p & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & q & 1-p-q & p & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & q & 1-p-q & p & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & q & 1-p-q & p & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & q & 1-p-q & p & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & q & 1-p-q & p & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & q & 1-p-q & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\boldsymbol{p} = (\ldots, 0, 0, 0, 0, 1, 0, 0, 0, 0, \ldots)$$

Markov chain - the birth and death process

(state = the population size)



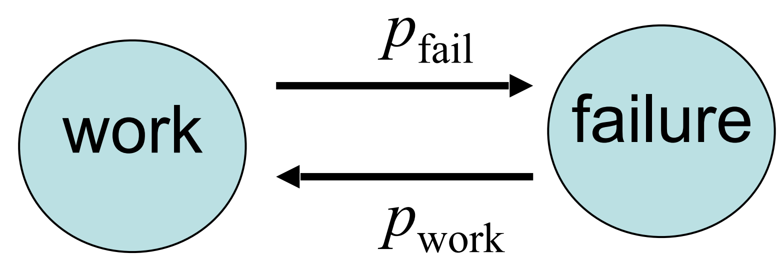


$$\mathbf{P} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1-p-q & p & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & q & 1-p-q & p & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & q & 1-p-q & p & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & q & 1-p-q & p & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & q & 1-p-q & p & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & q & 1-p-q & p & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & q & 1-p-q & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\boldsymbol{p} = (\ldots, 0, 0, 0, 0, 1, 0, 0, 0, 0, \ldots)$$

Markov chain - very simple homogeneous example

2-state unreliable system:

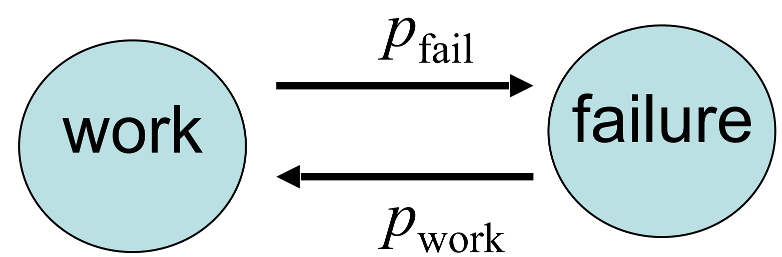


$\mathcal{S}$ = {„work“, „fail“}

$\boldsymbol{p} = \{1,0\}$

Markov chain - very simple homogeneous example

2-state unreliable system:



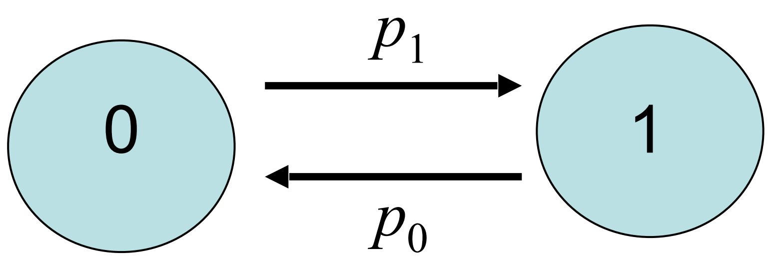
$\mathcal{S} = \{\text{„work“, „fail“}\}$

$\boldsymbol{p} = \{1{,}0\}$

$$\mathbf{P} = \begin{pmatrix} 1 - p_{\text{fail}} & p_{\text{fail}} \\ p_{\text{work}} & 1 - p_{\text{work}} \end{pmatrix}$$

Markov chain - very simple homogeneous example

2-state unreliable system:



$\mathcal{S} = \{0,1\}$

$\boldsymbol{p} = \{1,0\}$

$$\mathbf{P} = \begin{pmatrix} 1 - p_1 & p_1 \\ p_0 & 1 - p_0 \end{pmatrix}$$

Markov chain - let's go further!

$$p_{ij}(s,t) = P(X_t = j | X_s = i)$$

$$p_{ij}^{(n)} = P(X_{t+n} = j \mid X_t = i) = P(X_n = j \mid X_0 = i)$$

Markov chain - let's go further!

$$p_{ij}(s,t) = P(X_t = j | X_s = i)$$

$$p_{ij}^{(n)} = P(X_{t+n} = j \mid X_t = i) = P(X_n = j \mid X_0 = i)$$

$$p_i^{(n)} = P(X_n = i)$$

the marginal distribution

Markov chain - let's go further!

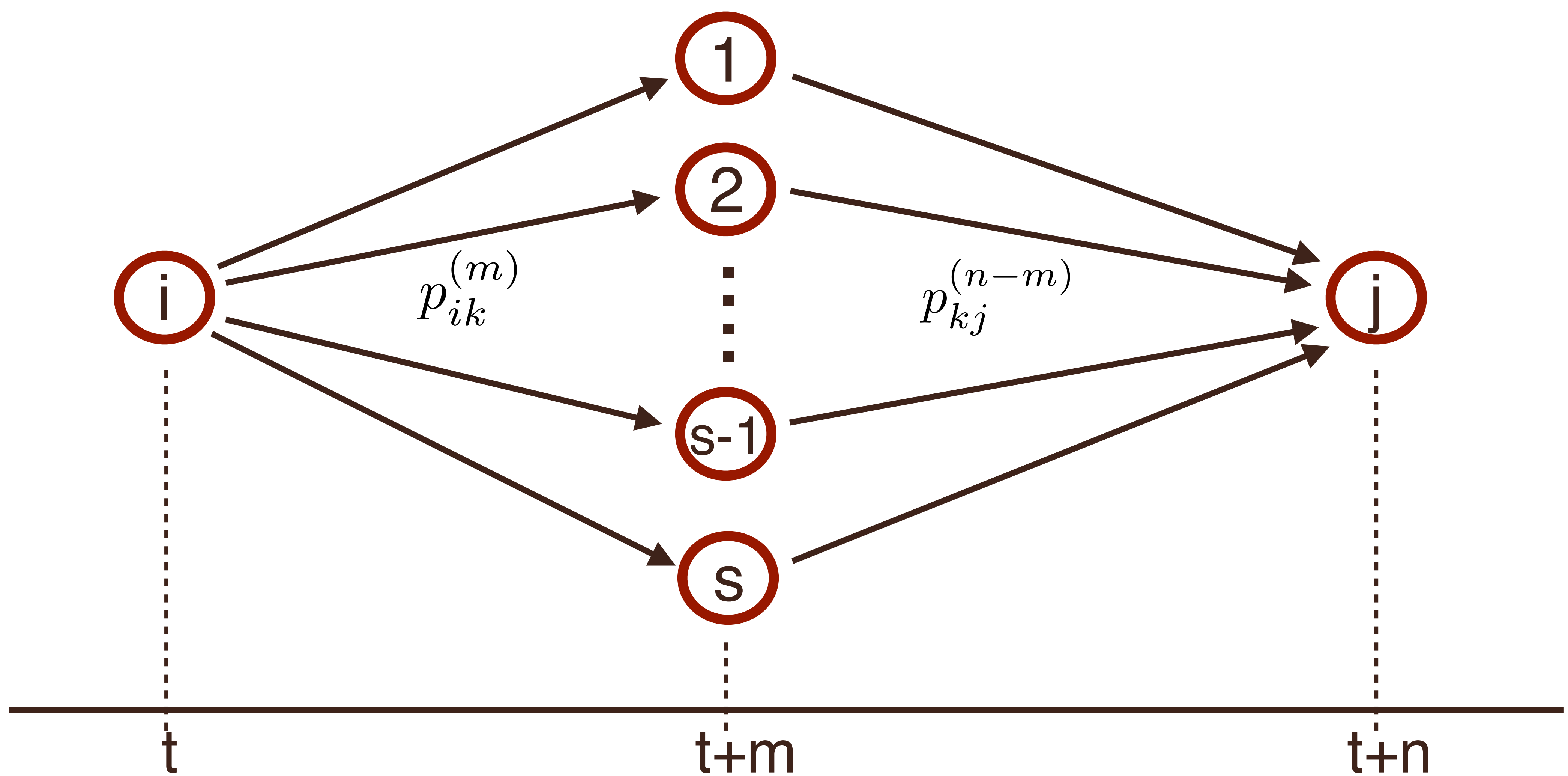
$$p_{ij}(s,t) = P(X_t = j | X_s = i)$$

$$p_{ij}^{(n)} = P(X_{t+n} = j \,|\, X_t = i) = P(X_n = j \,|\, X_0 = i)$$

$$p_i^{(n)} = P(X_n = i)$$

the marginal distribution

Theorem (Chapman-Kolmogorov formula):



Markov chain - let's go further!

$$p_{ij}(s,t) = P(X_t = j | X_s = i)$$

$$p_{ij}^{(n)} = P(X_{t+n} = j \,|\, X_t = i) = P(X_n = j \,|\, X_0 = i) \qquad p_i^{(n)} = P(X_n = i)$$

Theorem (Chapman-Kolmogorov formula):

Let us consider a homogeneous Markov chain $\{X_k\}_{k=1,2,\ldots}$ with finite set of states $\mathcal{S} = \{a_1, a_2, \ldots, a_s\}$ and a transition probability matrix $\mathbf{P}$. Then for any $0 < m < n$ and any two states i, j the following formula holds

$$p_{ij}^{(n)} = \sum_{k=1}^{s} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

Markov chain - let's go further!

$$p_{ij}(s,t) = P(X_t = j | X_s = i)$$

$$p_{ij}^{(n)} = P(X_{t+n} = j \,|\, X_t = i) = P(X_n = j \,|\, X_0 = i) \qquad p_i^{(n)} = P(X_n = i)$$

Theorem (Chapman-Kolmogorov formula):

Let us consider a homogeneous Markov chain $\{X_k\}_{k=1,2,\dots}$ with finite set of states $S = \{a_1, a_2, \dots, a_s\}$ and a transition probability matrix **P**. Then for any $0 < m < n$ and any two states i,j the following formula holds

$$p_{ij}^{(n)} = \sum_{k=1}^{s} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

for the marginal distribution: $p_i^{(n)} = \sum_{k=1}^{s} p_k^{(n-1)} p_{ki}$, or in matrix form: $\vec{p}^{(n)} = \vec{p}^{(n-1)}.P$

Markov chain - let's go further!

$$p_{ij}(s,t) = P(X_t = j | X_s = i)$$

$$p_{ij}^{(n)} = P(X_{t+n} = j \,|\, X_t = i) = P(X_n = j \,|\, X_0 = i) \qquad p_i^{(n)} = P(X_n = i)$$

Theorem (Chapman-Kolmogorov formula):

Let us consider a homogeneous Markov chain $\{X_k\}_{k=1,2,\ldots}$ with finite set of states $\mathcal{S} = \{a_1, a_2, \ldots, a_s\}$ and a transition probability matrix **P**. Then for any $0 < m < n$ and any two states i,j the following formula holds

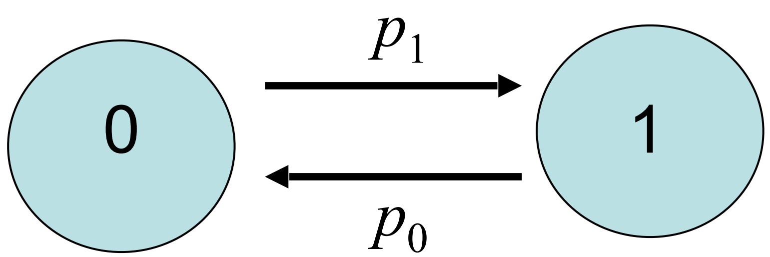
$$p_{ij}^{(n)} = \sum_{k=1}^{s} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

for the marginal distribution: $p_i^{(n)} = \sum_{k=1}^{s} p_k^{(n-1)} p_{ki}$, or in matrix form: $\vec{p}^{(n)} = \vec{p}^{(n-1)}.P$

If there exists $\lim_{n\to+\infty} \vec{p}^{(n)} = \vec{\pi}$ (stationary distribution) then $\vec{\pi} = \vec{\pi}.P$

Markov chain - very simple homogeneous example

2-state unreliable system:

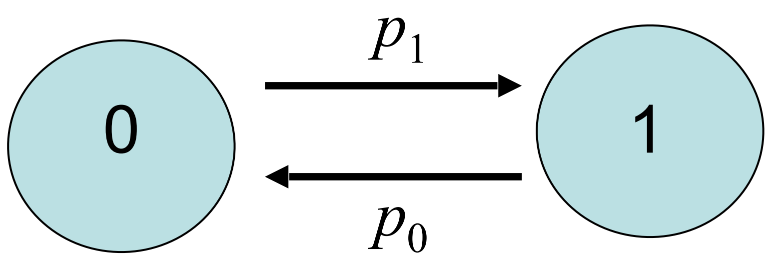


$\mathcal{S} = \{0,1\}$
$\boldsymbol{p} = \{1,0\}$

$$\mathbf{P} = \begin{pmatrix} 1 - p_1 & p_1 \\ p_0 & 1 - p_0 \end{pmatrix}$$

Markov chain - very simple homogeneous example

2-state unreliable system:



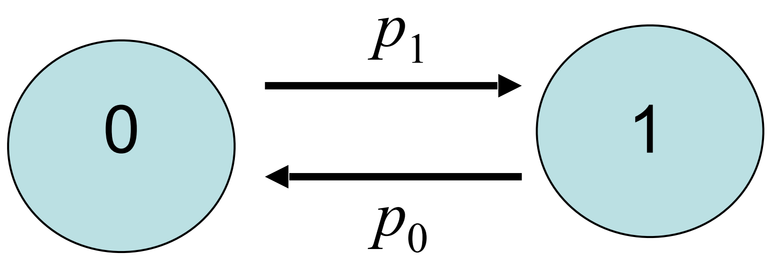
$$\mathcal{S} = \{0,1\}$$
$$\boldsymbol{p} = \{1,0\}$$

$$\mathbf{P} = \begin{pmatrix} 1-p_1 & p_1 \\ p_0 & 1-p_0 \end{pmatrix}$$

$$\vec{\pi} = \vec{\pi}.P$$

Markov chain - very simple homogeneous example

2-state unreliable system:



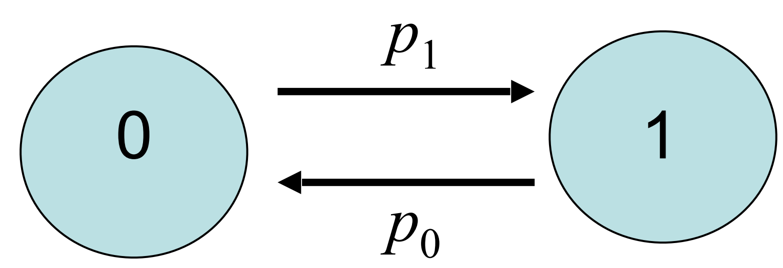
$$\mathcal{S} = \{0,1\}$$
$$\boldsymbol{p} = \{1,0\}$$

$$\mathbf{P} = \begin{pmatrix} 1-p_1 & p_1 \\ p_0 & 1-p_0 \end{pmatrix}$$

$$\vec{\pi} = \vec{\pi}.P$$
$$P^T \vec{\pi}^T = \vec{\pi}^T$$

Markov chain - very simple homogeneous example

2-state unreliable system:



$$\mathcal{S} = \{0,1\}$$
$$\boldsymbol{p} = \{1,0\}$$

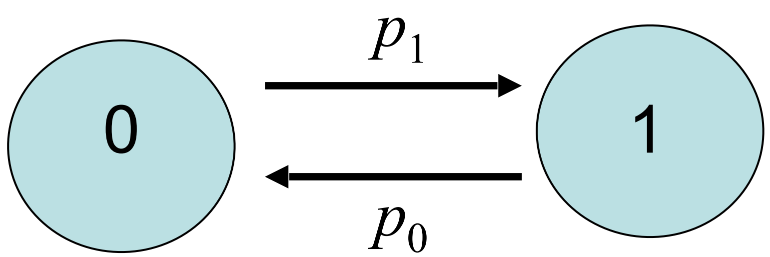
$$\mathbf{P} = \begin{pmatrix} 1 - p_1 & p_1 \\ p_0 & 1 - p_0 \end{pmatrix}$$

$$\begin{aligned} \vec{\pi} &= \vec{\pi}.P \\ P^T \vec{\pi}^T &= \vec{\pi}^T \\ (P^T - I)\vec{\pi}^T &= \vec{0} \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

$$\begin{pmatrix} -p_1 & p_0 \\ p_1 & -p_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Markov chain - very simple homogeneous example

2-state unreliable system:



$\mathcal{S} = \{0,1\}$

$\boldsymbol{p} = \{1,0\}$

$$\mathbf{P} = \begin{pmatrix} 1-p_1 & p_1 \\ p_0 & 1-p_0 \end{pmatrix}$$

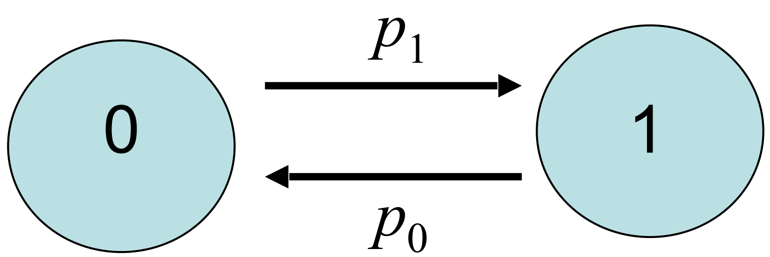
$$\begin{aligned} \vec{\pi} &= \vec{\pi}.P \\ P^T \vec{\pi}^T &= \vec{\pi}^T \\ (P^T - I)\vec{\pi}^T &= \vec{0} \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

$$\begin{pmatrix} -p_1 & p_0 \\ p_1 & -p_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\pi_0 = \frac{p_0}{p_0 + p_1}, \quad \pi_1 = \frac{p_1}{p_0 + p_1}$$

Markov chain - very simple homogeneous example

2-state unreliable system:



$\mathcal{S} = \{0,1\}$

$\boldsymbol{p} = \{1,0\}$

$$\mathbf{P} = \begin{pmatrix} 1-p_1 & p_1 \\ p_0 & 1-p_0 \end{pmatrix}$$

$$\vec{\pi} = \vec{\pi}.P$$

$$P^T\vec{\pi}^T = \vec{\pi}^T$$

$$(P^T - I)\vec{\pi}^T = \vec{0}$$

$$\pi_0 + \pi_1 = 1$$

$$\begin{pmatrix} -p_1 & p_0 \\ p_1 & -p_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\pi_0 = \frac{p_0}{p_0+p_1}, \quad \pi_1 = \frac{p_1}{p_0+p_1}$$

asymptotic availability asymptotic unavailability

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- $p_{ij}(s,t) \in [0,1]$, $\mathrm{P(s,t)} \geq \mathbf{0}$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- $p_{ij}(s,t) \in [0,1]$, $\mathrm{P}(\mathrm{s,t}) \geq \mathbf{0}$

- $p_{ii}(t,t) = 1$, $p_{ij}(t,t) = 0$, $\mathrm{P}(\mathrm{t,t}) \geq \mathbf{I}$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- $p_{ij}(s,t) \in [0,1]$, $\mathrm{P(s,t)} \geq \mathbf{0}$
- $p_{ii}(t,t) = 1, p_{ij}(t,t) = 0$, $\mathrm{P(t,t)} \geq \mathbf{I}$
- $\sum_{j \in \mathcal{S}} p_{ij}(s,t) = 1$, $\mathrm{P(s,t)}.\mathbf{e}=\mathbf{e}$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- $p_{ij}(s,t) \in [0,1]$, $\mathrm{P}(\mathrm{s,t}) \geq \mathbf{0}$
- $p_{ii}(t,t) = 1, p_{ij}(t,t) = 0$, $\mathrm{P}(\mathrm{t,t}) \geq \mathbf{I}$
- $\sum_{j \in \mathcal{S}} p_{ij}(s,t) = 1$, $\mathrm{P}(\mathrm{s,t}).\mathbf{e}=\mathbf{e}$
- Chapman-Kolmogorov equation:

$$p_{ij}(s, s+t+h) = \sum_{k \in \mathcal{S}} p_{ik}(s, s+t) p_{kj}(s+t, s+t+h)$$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- $p_{ij}(s,t) \in [0,1]$, $\mathrm{P}(s,t) \geq \mathbf{0}$

- $p_{ii}(t,t) = 1$, $p_{ij}(t,t) = 0$, $\mathrm{P}(t,t) \geq \mathbf{I}$

- $\sum_{j \in \mathcal{S}} p_{ij}(s,t) = 1$, $\mathrm{P}(s,t).\mathbf{e}=\mathbf{e}$

- Chapman-Kolmogorov equation:

$$p_{ij}(s, s+t+h) = \sum_{k \in \mathcal{S}} p_{ik}(s, s+t) p_{kj}(s+t, s+t+h)$$

$$\mathrm{P}(s,s+t+h) = \mathrm{P}(s,s+t).\mathrm{P}(s+t,s+t+h)$$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- $p_{ij}(s,t) \in [0,1]$, $\mathrm{P}(s,t) \geq \mathbf{0}$
- $p_{ii}(t,t) = 1$, $p_{ij}(t,t) = 0$, $\mathrm{P}(t,t) \geq \mathbf{I}$
- $\sum_{j \in \mathcal{S}} p_{ij}(s,t) = 1$, $\mathrm{P}(s,t).\mathbf{e}=\mathbf{e}$
- Chapman-Kolmogorov equation:

$$p_{ij}(s, s+t+h) = \sum_{k \in \mathcal{S}} p_{ik}(s, s+t) p_{kj}(s+t, s+t+h)$$

$$\mathrm{P}(s,s+t+h) = \mathrm{P}(s,s+t).\mathrm{P}(s+t,s+t+h)$$

- evolution law: $p_j(s+t) = \sum_{k \in \mathcal{S}} p_k(s) p_{kj}(s, s+t)$

$$\mathbf{p}(s+t) = \mathbf{p}(s).\mathrm{P}(s,s+t)$$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Homogeneous process: $p_{ij}(s,t) = p_{ij}(t\text{-}s)$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Homogeneous process: $p_{ij}(s,t) = p_{ij}(t\text{-}s)$
- Transition intensities: $q_{ii} = \lim_{h \to 0+} \frac{p_{ii}(h) - 1}{h} \quad q_{ij} = \lim_{h \to 0+} \frac{p_{ij}(h)}{h}$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Homogeneous process: $p_{ij}(s,t) = p_{ij}(t\text{-}s)$
- Transition intensities: $q_{ii} = \lim_{h \to 0+} \frac{p_{ii}(h) - 1}{h}$ $q_{ij} = \lim_{h \to 0+} \frac{p_{ij}(h)}{h}$
- q_{ii} = intensity of persistence in the state s_i

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Homogeneous process: $p_{ij}(s,t) = p_{ij}(t\text{-}s)$
- Transition intensities: $q_{ii} = \lim\limits_{h \to 0+} \dfrac{p_{ii}(h) - 1}{h}$ $\quad q_{ij} = \lim\limits_{h \to 0+} \dfrac{p_{ij}(h)}{h}$
- q_{ii} = intensity of persistence in the state s_i
- the distribution of persistence in the state s_i ~ Exp($-q_{ii}$)

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Homogeneous process: $p_{ij}(s,t) = p_{ij}(t\text{-}s)$
- Transition intensities: $q_{ii} = \lim_{h \to 0+} \dfrac{p_{ii}(h) - 1}{h}$ $\quad q_{ij} = \lim_{h \to 0+} \dfrac{p_{ij}(h)}{h}$
- q_{ii} = intensity of persistence in the state s_i
- the distribution of persistence in the state s_i ~ Exp($-q_{ii}$)
- the transition intenzities matrix:

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Homogeneous process: $p_{ij}(s,t) = p_{ij}(t\text{-}s)$
- Transition intensities: $q_{ii} = \lim_{h \to 0+} \frac{p_{ii}(h) - 1}{h}$ $q_{ij} = \lim_{h \to 0+} \frac{p_{ij}(h)}{h}$
- q_{ii} = intensity of persistence in the state s_i
- the distribution of persistence in the state s_i ~ Exp($-q_{ii}$)
- the transition intenzities matrix:

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

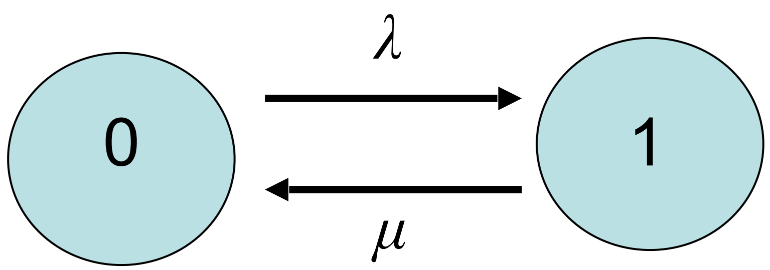
- the system of Kolmogorov differential equations:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\vec{p}(t) = \vec{p}(t) \cdot Q}$$

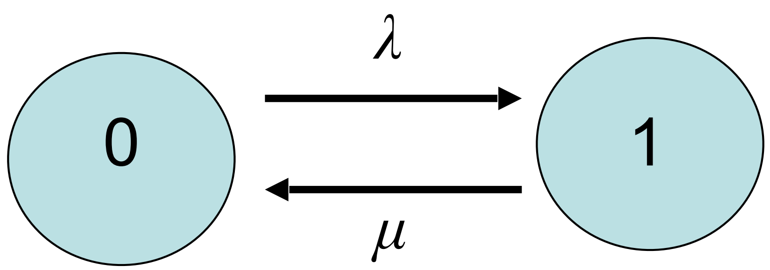
Markov process - very simple homogeneous example

2-state unreliable system:

$S = \{0,1\}$



Markov process - very simple homogeneous example

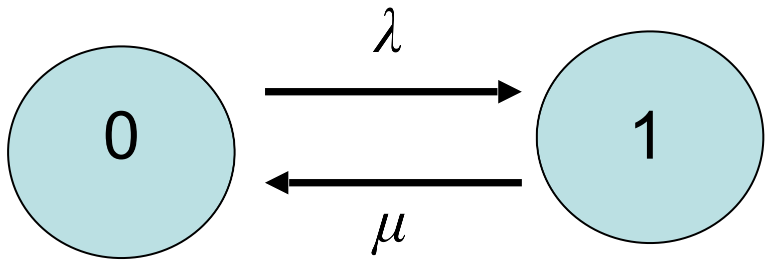


2-state unreliable system:

$S = \{0,1\}$

$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$

Markov process - very simple homogeneous example



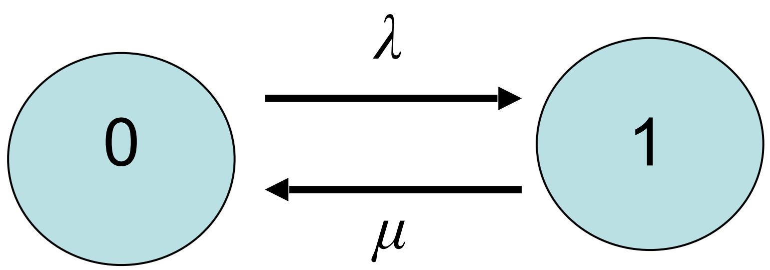
2-state unreliable system:

$\mathcal{S} = \{0,1\}$

$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$

$$p_{10}(h) = 1 - e^{-\mu h} = \mu h + o(h)$$

Markov process - very simple homogeneous example



2-state unreliable system:

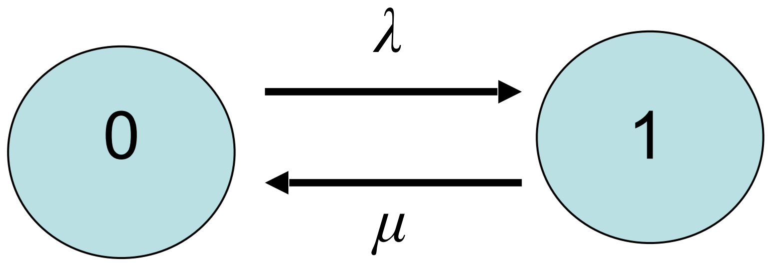
$\mathcal{S} = \{0,1\}$

$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$

$$p_{10}(h) = 1 - e^{-\mu h} = \mu h + o(h)$$

$$p_{00}(h) = e^{-\lambda h} = 1 - \lambda h + o(h)$$

Markov process - very simple homogeneous example



2-state unreliable system:

$S = \{0,1\}$

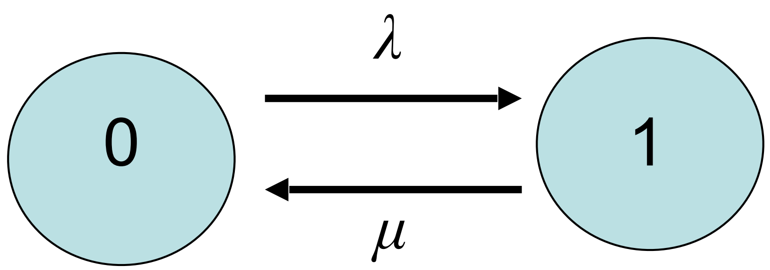
$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$

$$p_{10}(h) = 1 - e^{-\mu h} = \mu h + o(h)$$

$$p_{00}(h) = e^{-\lambda h} = 1 - \lambda h + o(h)$$

$$p_{11}(h) = e^{-\mu h} = 1 - \mu h + o(h)$$

Markov process - very simple homogeneous example



2-state unreliable system:

$\mathcal{S} = \{0,1\}$

$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$

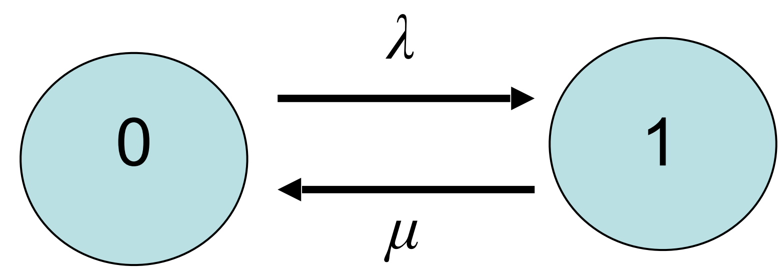
$$p_{10}(h) = 1 - e^{-\mu h} = \mu h + o(h)$$

$$p_{00}(h) = e^{-\lambda h} = 1 - \lambda h + o(h)$$

$$p_{11}(h) = e^{-\mu h} = 1 - \mu h + o(h)$$

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

Markov process - very simple homogeneous example



2-state unreliable system:

$$S = \{0,1\}$$

$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$

$$p_{10}(h) = 1 - e^{-\mu h} = \mu h + o(h)$$

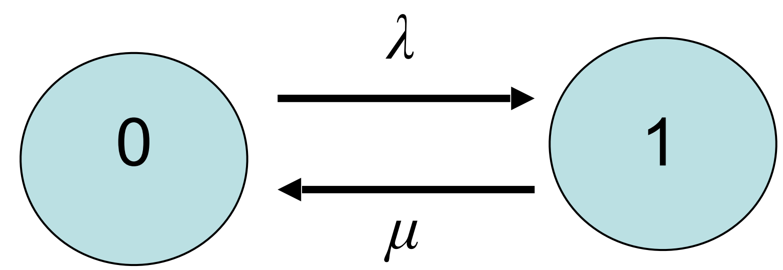
$$p_{00}(h) = e^{-\lambda h} = 1 - \lambda h + o(h)$$

$$p_{11}(h) = e^{-\mu h} = 1 - \mu h + o(h)$$

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$\boxed{\begin{gathered} \lim_{t\to\infty} \vec{p}(t) = \vec{\pi} \\ 0 = \vec{\pi} \cdot Q \end{gathered}}$$

Markov process - very simple homogeneous example



2-state unreliable system:

$$S = \{0,1\}$$

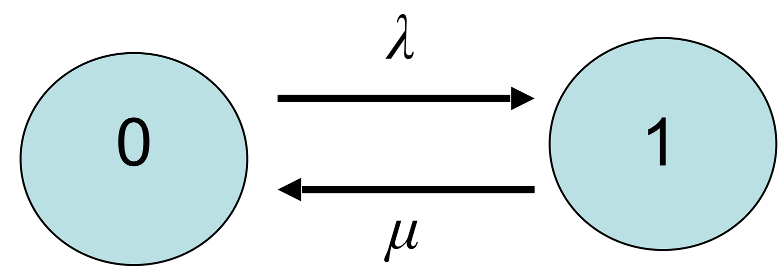
$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$
$$p_{10}(h) = 1 - e^{-\mu h} = \mu h + o(h)$$
$$p_{00}(h) = e^{-\lambda h} = 1 - \lambda h + o(h)$$
$$p_{11}(h) = e^{-\mu h} = 1 - \mu h + o(h)$$

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = -\lambda p_1(t) + \mu p_2(t)$$
$$\frac{\mathrm{d}p_2(t)}{\mathrm{d}t} = \lambda p_1(t) - \mu p_2(t)$$
$$p_1(0) = 1, \qquad p_2(0) = 0$$

$$\boxed{\begin{array}{c} \lim_{t \to \infty} \vec{p}(t) = \vec{\pi} \\ 0 = \vec{\pi} \cdot Q \end{array}}$$

Markov process - very simple homogeneous example



2-state unreliable system:

$$\mathcal{S} = \{0,1\}$$

$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$
$$p_{10}(h) = 1 - e^{-\mu h} = \mu h + o(h)$$
$$p_{00}(h) = e^{-\lambda h} = 1 - \lambda h + o(h)$$
$$p_{11}(h) = e^{-\mu h} = 1 - \mu h + o(h)$$

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

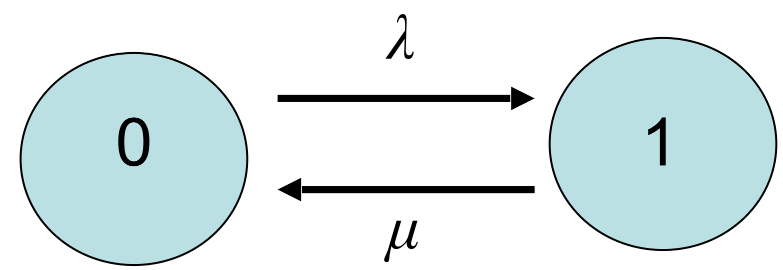
$$\lim_{t\to\infty} \vec{p}(t) = \vec{\pi}$$
$$0 = \vec{\pi} \cdot Q$$

$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = -\lambda p_1(t) + \mu p_2(t)$$
$$\frac{\mathrm{d}p_2(t)}{\mathrm{d}t} = \lambda p_1(t) - \mu p_2(t)$$
$$p_1(0) = 1, \qquad p_2(0) = 0$$

$$p_1(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$
$$p_2(t) = \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right)$$

Markov process - very simple homogeneous example

2-state unreliable system:



$\mathcal{S} = \{0,1\}$

$$p_{01}(h) = 1 - e^{-\lambda h} = \lambda h + o(h)$$
$$p_{10}(h) = 1 - e^{-\mu h} = \mu h + o(h)$$
$$p_{00}(h) = e^{-\lambda h} = 1 - \lambda h + o(h)$$
$$p_{11}(h) = e^{-\mu h} = 1 - \mu h + o(h)$$

$$\frac{dp_1(t)}{dt} = -\lambda p_1(t) + \mu p_2(t)$$
$$\frac{dp_2(t)}{dt} = \lambda p_1(t) - \mu p_2(t)$$
$$p_1(0) = 1, \qquad p_2(0) = 0$$

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

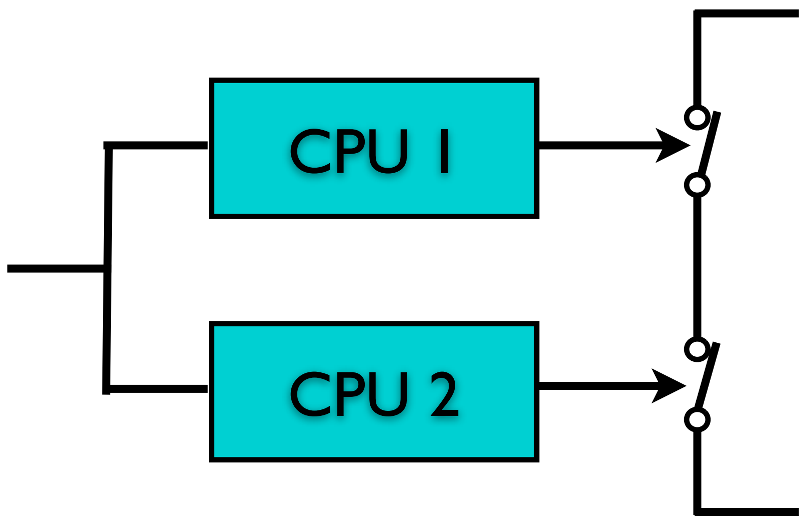
$$\boxed{\begin{gathered} \lim_{t\to\infty} \vec{p}(t) = \vec{\pi} \\ 0 = \vec{\pi} \cdot Q \end{gathered}}$$

$$\boxed{\begin{gathered} p_1(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \\ p_2(t) = \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right) \end{gathered}}$$

$$\lim_{t\to\infty} p_1(t) = \frac{\mu}{\lambda+\mu}, \qquad \lim_{t\to\infty} p_2(t) = \frac{\lambda}{\lambda+\mu}$$

Markov process - a bit more complex homogeneous example

Logical unit in a protection system

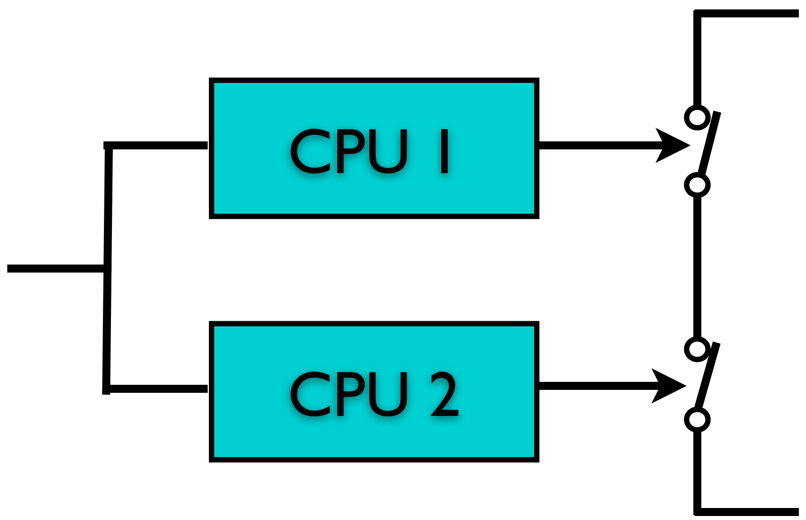


Markov process - a bit more complex homogeneous example

Logical unit in a protection system

S = {WW, WFR, WFU, SF, DF}

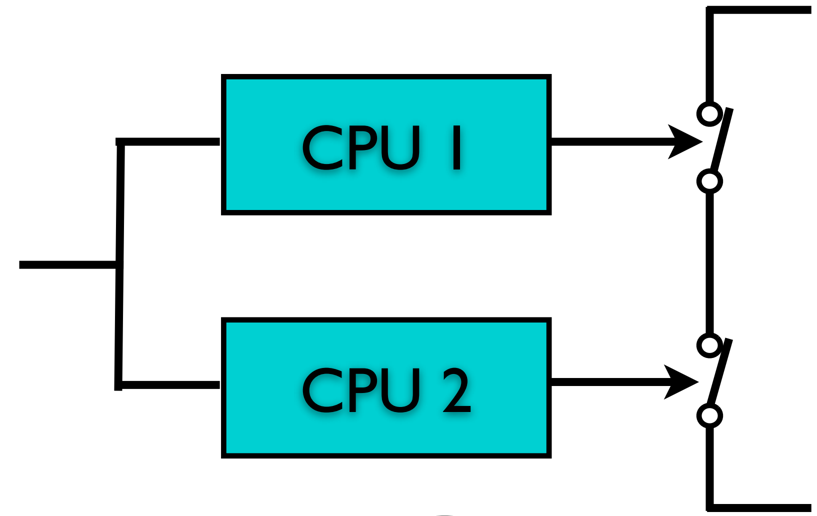
WW = well working system
WFR = partially working system with recognizable failure
WFU = partially working system with unrecognizable failure
SF = system in safe failure
DF = system in dangerous failure



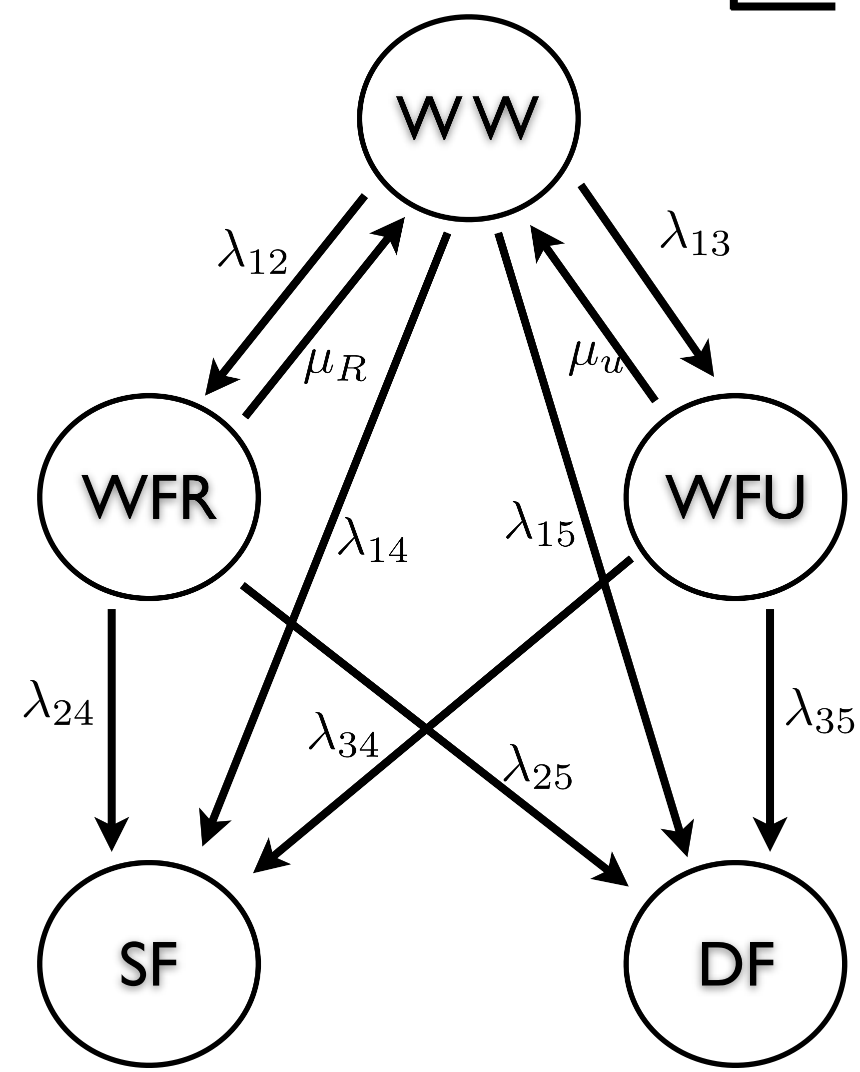
Markov process - a bit more complex homogeneous example

Logical unit in a protection system

$S = \{WW, WFR, WFU, SF, DF\}$



WW = well working system
WFR = partially working system with recognizable failure
WFU = partially working system with unrecognizable failure
SF = system in safe failure
DF = system in dangerous failure

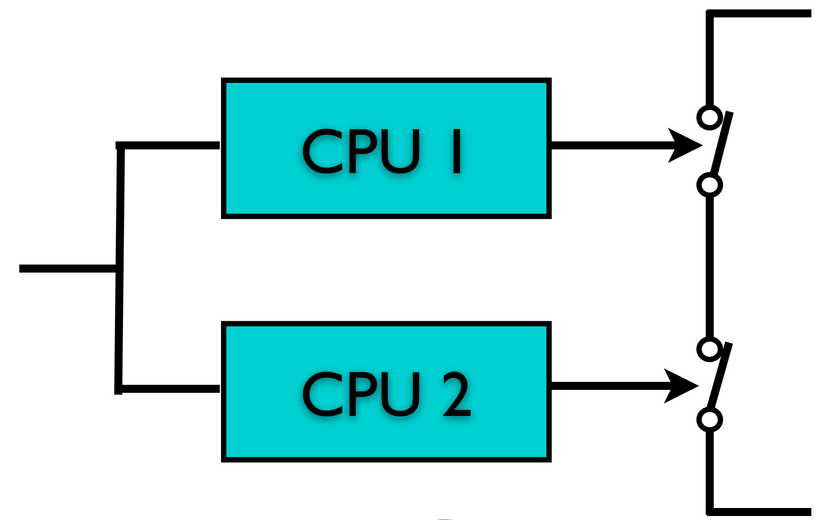


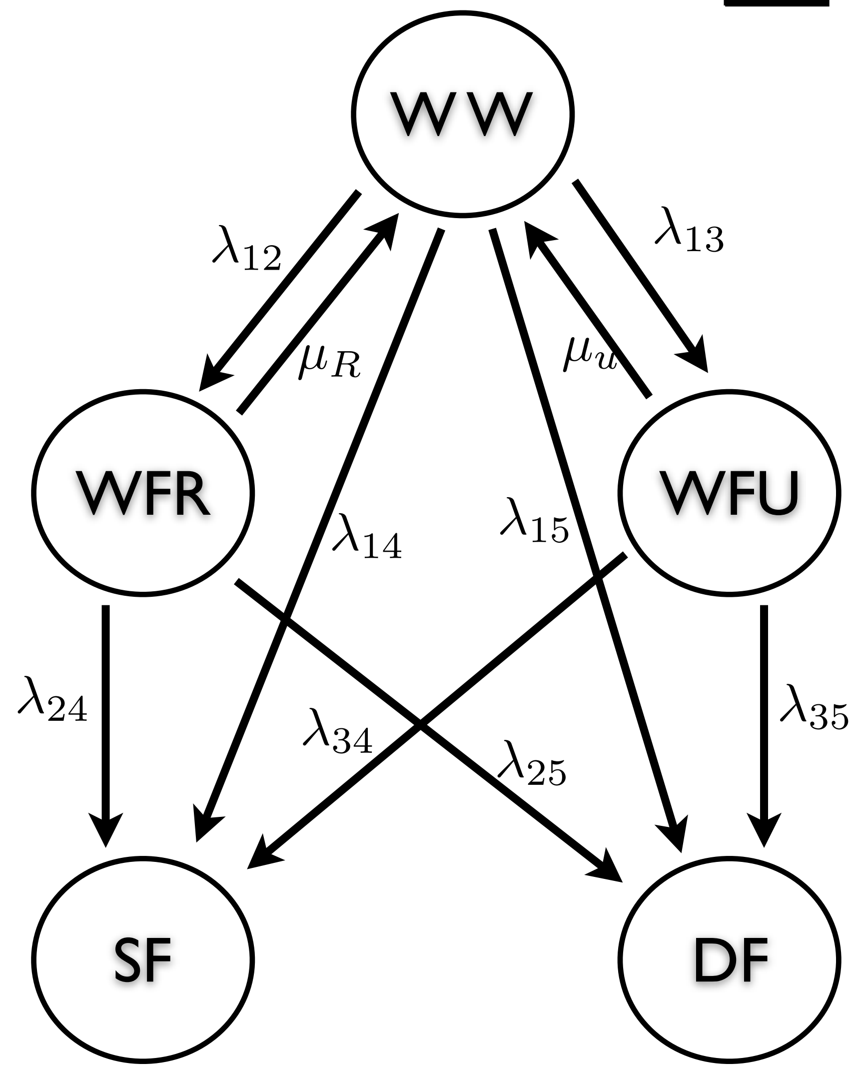
Markov process - a bit more complex homogeneous example

Logical unit in a protection system

$\mathcal{S} = \{\text{WW, WFR, WFU, SF, DF}\}$

WW = well working system
WFR = partially working system with recognizable failure
WFU = partially working system with unrecognizable failure
SF = system in safe failure
DF = system in dangerous failure





$$Q = \begin{pmatrix} q_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \mu_R & q_{22} & 0 & \lambda_{24} & \lambda_{25} \\ \mu_U & 0 & q_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\vec{a} = (1, 0, 0, 0, 0)$$

Markov process - a bit more complex homogeneous example

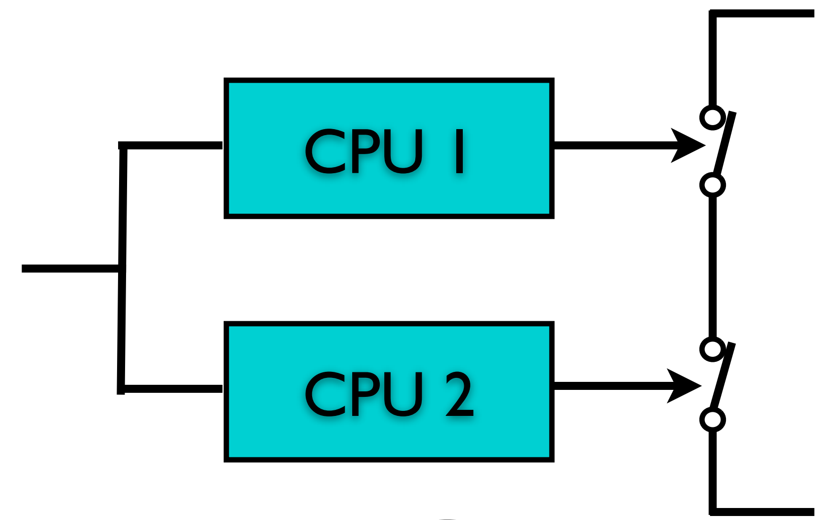
Logical unit in a protection system

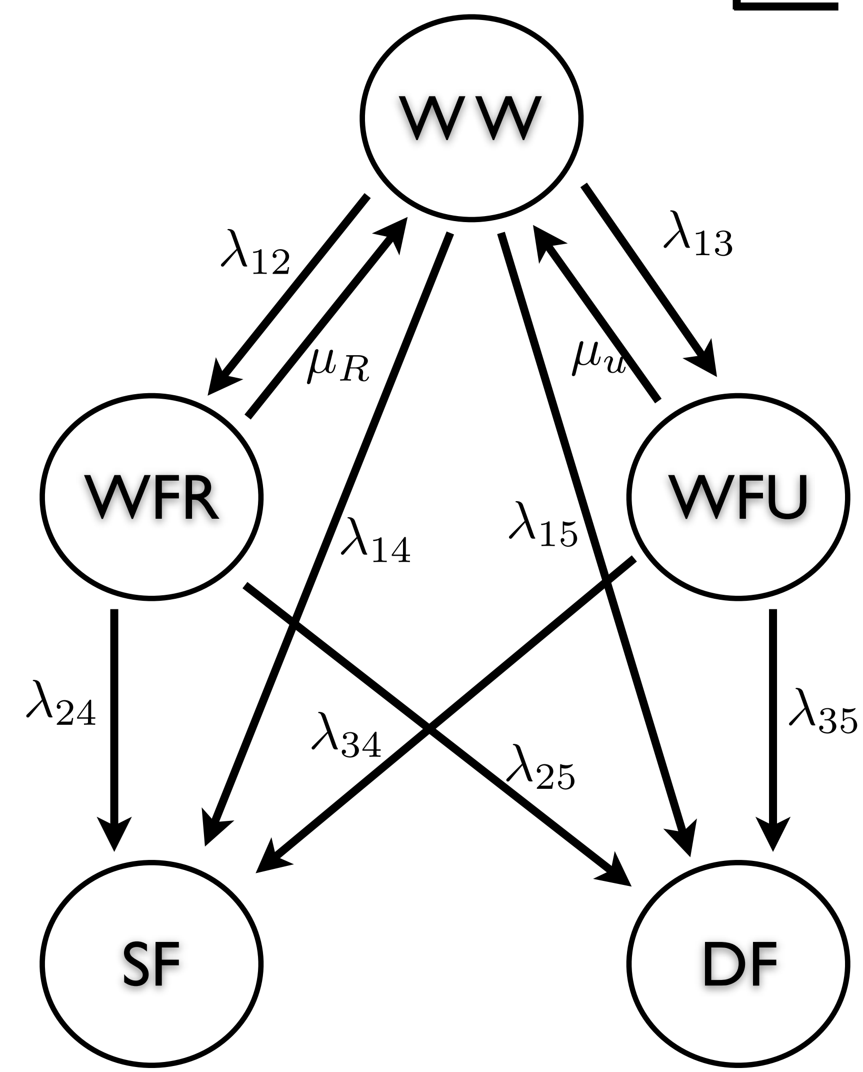
$S = \{WW, WFR, WFU, SF, DF\}$

WW = well working system
WFR = partially working system with recognizable failure
WFU = partially working system with unrecognizable failure
SF = system in safe failure
DF = system in dangerous failure

$$Q = \begin{pmatrix} q_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \mu_R & q_{22} & 0 & \lambda_{24} & \lambda_{25} \\ \mu_U & 0 & q_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\vec{a} = (1, 0, 0, 0, 0)$$





Markov process - a bit more complex homogeneous example

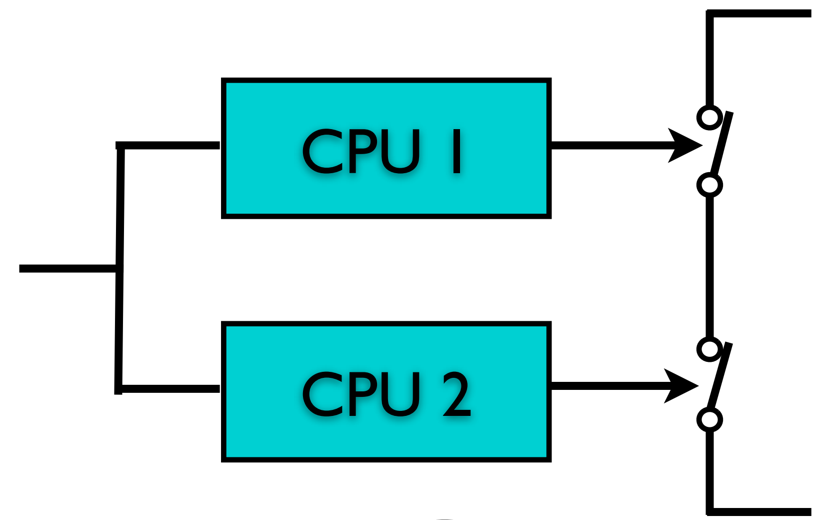
Logical unit in a protection system

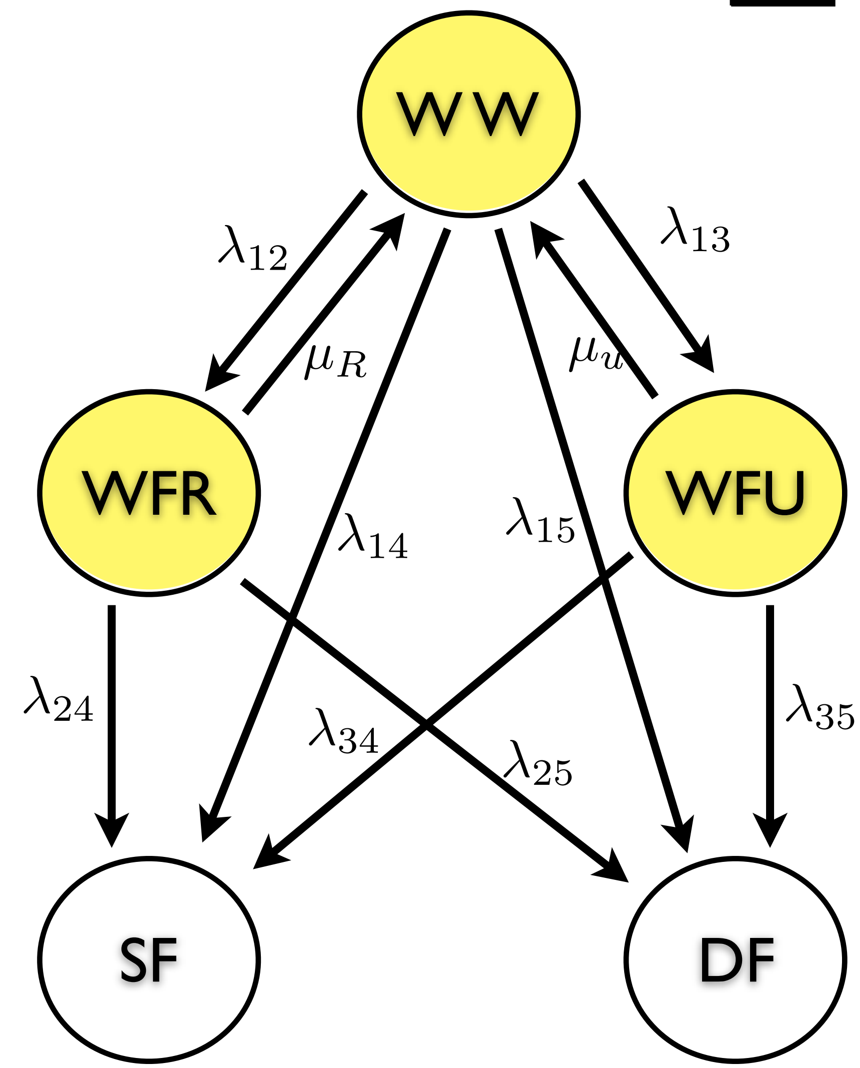
S = {WW, WFR, WFU, SF, DF}

WW = well working system
WFR = partially working system with recognizable failure
WFU = partially working system with unrecognizable failure
SF = system in safe failure
DF = system in dangerous failure

$$Q = \begin{pmatrix} q_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \mu_R & q_{22} & 0 & \lambda_{24} & \lambda_{25} \\ \mu_U & 0 & q_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

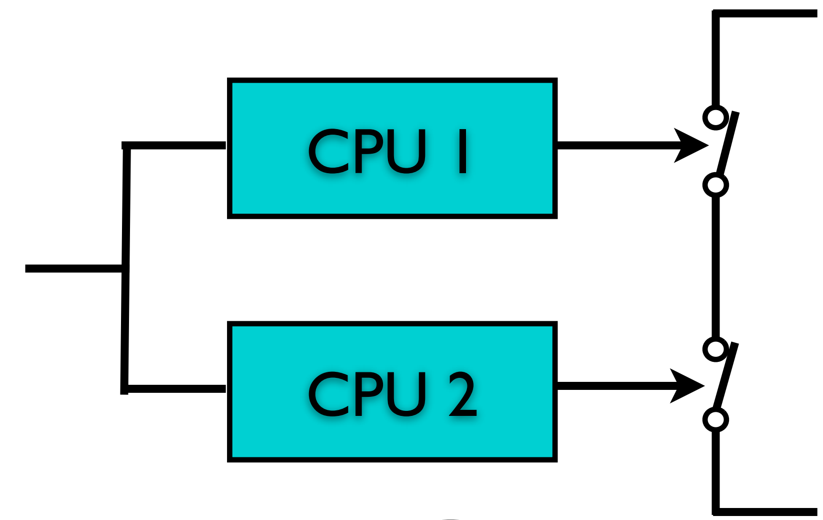
$$\vec{a} = (1, 0, 0, 0, 0)$$





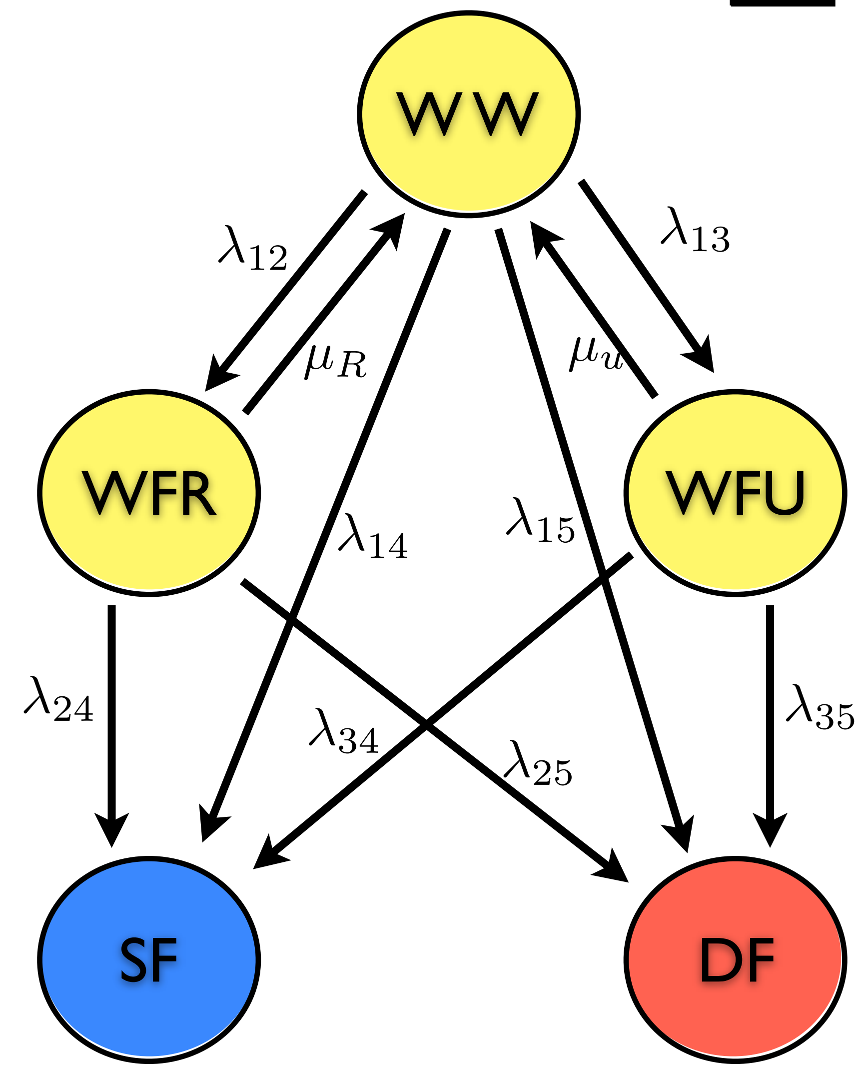
Markov process - a bit more complex homogeneous example

Logical unit in a protection system



S = {WW, WFR, WFU, SF, DF}

WW = well working system
WFR = partially working system with recognizable failure
WFU = partially working system with unrecognizable failure
SF = system in safe failure
DF = system in dangerous failure



$$Q = \begin{pmatrix} q_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \mu_R & q_{22} & 0 & \lambda_{24} & \lambda_{25} \\ \mu_U & 0 & q_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

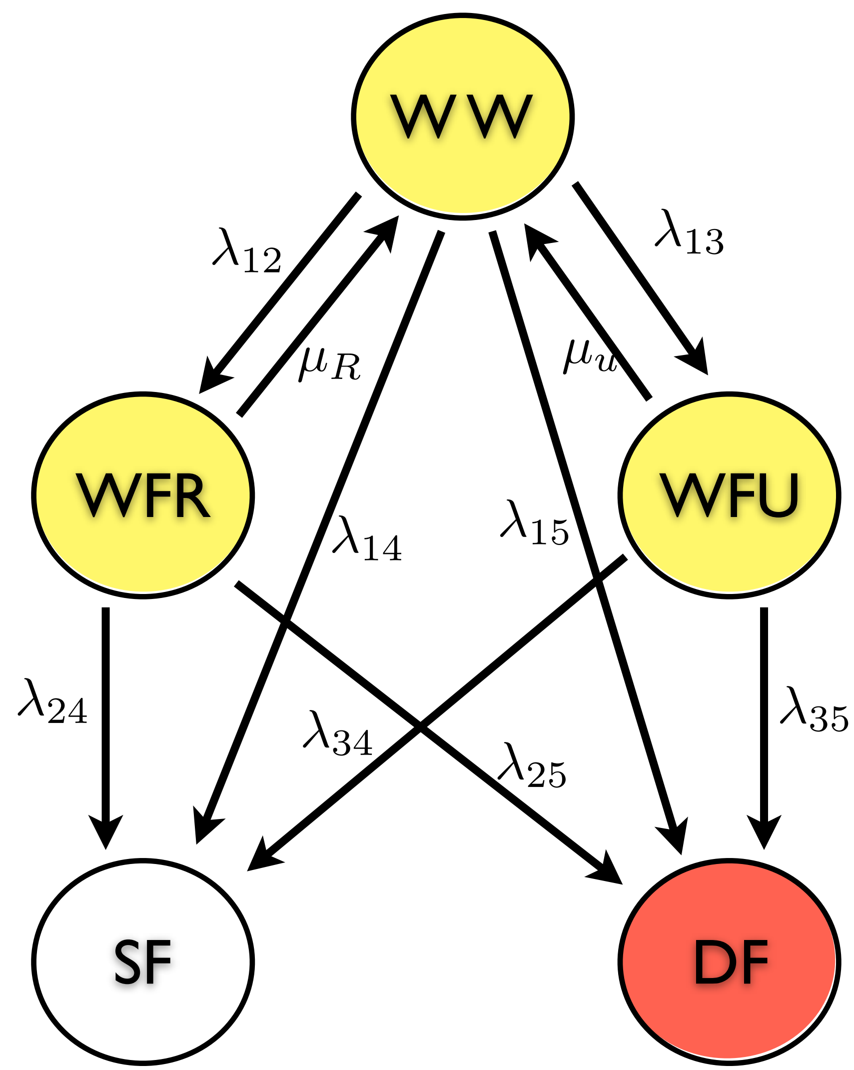
$$\vec{a} = (1, 0, 0, 0, 0)$$

Markov process - a bit more complex homogeneous example

Logical unit in a protection system

$$\mathcal{S} = \{\mathrm{WW}, \mathrm{WFR}, \mathrm{WFU}, \mathrm{SF}, \mathrm{DF}\}$$

$$Q^* = \begin{pmatrix} q^*_{11} & \lambda_{12} & \lambda_{13} & \lambda_{15} \\ \mu_R & q^*_{22} & 0 & \lambda_{25} \\ \mu_U & 0 & q^*_{33} & \lambda_{35} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



$$Q = \begin{pmatrix} q_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \mu_R & q_{22} & 0 & \lambda_{24} & \lambda_{25} \\ \mu_U & 0 & q_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\vec{a} = (1, 0, 0, 0, 0)$$

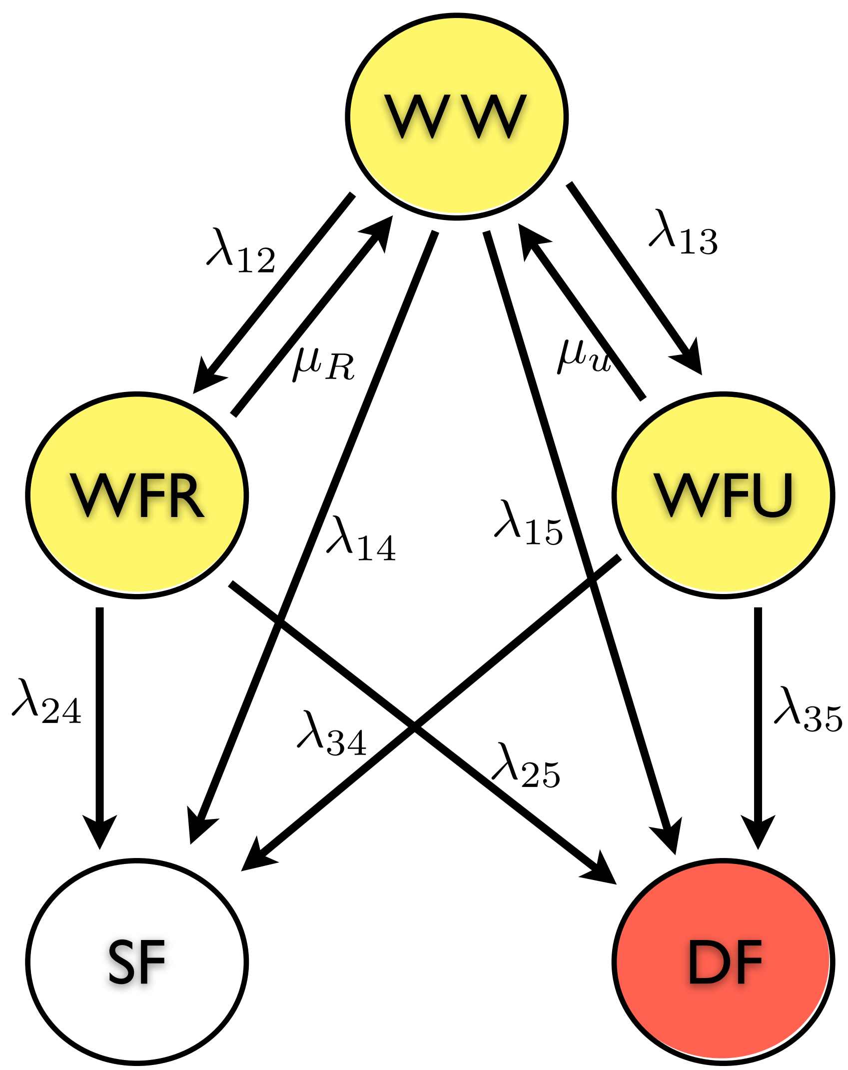
Markov process - a bit more complex homogeneous example

Logical unit in a protection system

$\mathcal{S} = \{\text{WW, WFR, WFU, SF, DF}\}$

Q_{DF} $\vec{p}_{DF}$

$$Q^* = \begin{pmatrix} q^*_{11} & \lambda_{12} & \lambda_{13} & \lambda_{15} \\ \mu_R & q^*_{22} & 0 & \lambda_{25} \\ \mu_U & 0 & q^*_{33} & \lambda_{35} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



$$Q = \begin{pmatrix} q_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \mu_R & q_{22} & 0 & \lambda_{24} & \lambda_{25} \\ \mu_U & 0 & q_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\vec{a} = (1, 0, 0, 0, 0)$

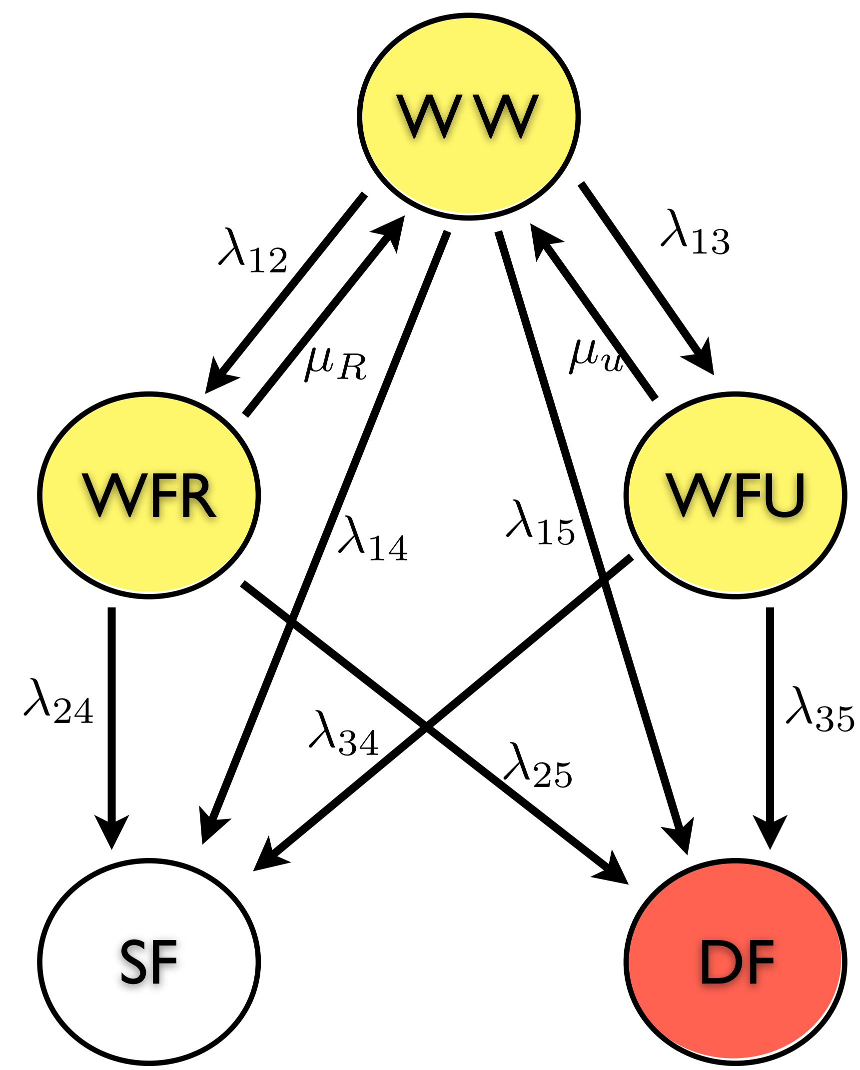
Markov process - a bit more complex homogeneous example

Logical unit in a protection system

$\mathcal{S} = \{\mathrm{WW}, \mathrm{WFR}, \mathrm{WFU}, \mathrm{SF}, \mathrm{DF}\}$

$$\begin{array}{cc} & \begin{array}{cc} \qquad Q_{DF} \qquad\qquad & \vec{p}_{DF} \end{array} \\ Q^* = & \begin{pmatrix} q^*_{11} & \lambda_{12} & \lambda_{13} & \lambda_{15} \\ \mu_R & q^*_{22} & 0 & \lambda_{25} \\ \mu_U & 0 & q^*_{33} & \lambda_{35} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\boxed{T_{DF} = -\vec{a} Q^{-1}_{DF} \vec{e}\,'}$$



$$Q = \begin{pmatrix} q_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \mu_R & q_{22} & 0 & \lambda_{24} & \lambda_{25} \\ \mu_U & 0 & q_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\vec{a} = (1, 0, 0, 0, 0)$

Continuous time

$\{X_t\}t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- Inhomogeneous process: $p_{ij}(s,t)$, transition intensities:

$$q_{ii}(t) = \lim_{h \to 0_+} \frac{p_{ii}(t, t+h) - 1}{h}, \quad q_{ij}(t) = \lim_{h \to 0_+} \frac{p_{ij}(t, t+h)}{h}$$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- Inhomogeneous process: $p_{ij}(s,t)$, transition intensities:

$$q_{ii}(t) = \lim_{h \to 0_+} \frac{p_{ii}(t, t+h) - 1}{h}, \quad q_{ij}(t) = \lim_{h \to 0_+} \frac{p_{ij}(t, t+h)}{h}$$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \dots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- Inhomogeneous process: $p_{ij}(s,t)$, transition intensities:

$$q_{ii}(t) = \lim_{h \to 0_+} \frac{p_{ii}(t, t+h) - 1}{h}, \quad q_{ij}(t) = \lim_{h \to 0_+} \frac{p_{ij}(t, t+h)}{h}$$

- $q_{ii}(t)$ = intensity of persistence in the state s_i, the distribution of persistence in the state: $p_{ii}(t, t+h) = \exp\big(- \int_0^h \sum_{j \neq i} q_{ij}(t+s) \mathrm{d}s \big)$

Continuous time

$\{X_t\} t \in \mathbf{R}$

- Markov property in continuous time:

$$\boxed{P(X_t = i \mid X_s = j) = P(X_t = i \mid X_s = j, X_{s_1} = j_1, \ldots, X_{s_k} = j_k) = p_{ij}(s,t)}$$

- Inhomogeneous process: $p_{ij}(s,t)$, transition intensities:

$$q_{ii}(t) = \lim_{h \to 0_+} \frac{p_{ii}(t,t+h) - 1}{h}, \quad q_{ij}(t) = \lim_{h \to 0_+} \frac{p_{ij}(t,t+h)}{h}$$

- $q_{ii}(t)$ = intensity of persistence in the state s_i, the distribution of persistence in the state: $p_{ii}(t,t+h) = \exp\big(-\int_0^h \sum_{j \neq i} q_{ij}(t+s)\mathrm{d}s\big)$

- the system of Kolmogorov differential equations:

$$P'(t,t+h) = P(t,t+h).Q(t+h)$$

Semi-Markov process

https://www.sciencedirect.com/topics/computer-science/semi-markov-process

Semi-Markov process

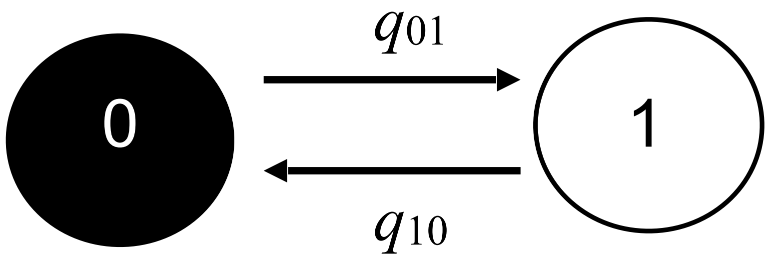
$\{X_t\} t \geq 0$

- Random process with values in $\mathcal{S}$ = {a$_1$, a_2, …, a_s} .
- Transitions between states occur in random times $t_n = \sum_{i=0}^{n-1} \tau_i, \; n = 1, 2, \ldots$ only.
- Transitions follow some Markov process in discrete time wit transition matrix **P** (nested Markov process).
- Let $F_{ij}(t)$ be a transition cdf between states i and j. Denote **H** the matrix of $F_{ij}(t)$.
- Semi-Markov process is given by the triple $(\mathbf{p}, \mathbf{P}, \mathbf{H})$.
- The process $\{(X_n, t_n)\}, \; n = 0, 1, 2, \ldots$ is homogeneous Markov process.
- Markov process in continuous time can be interpreted as a semi-Markov process with exponential persistence times.

https://www.sciencedirect.com/topics/computer-science/semi-markov-process

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):

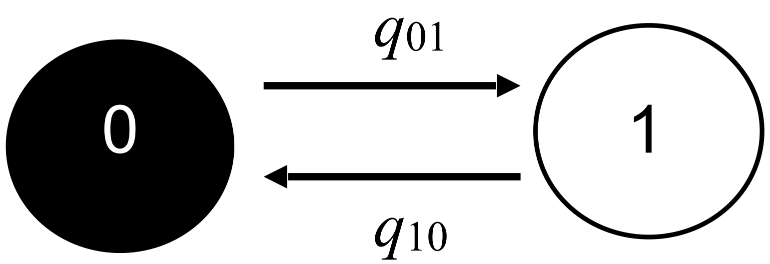


$$\vec{p} = (p_0, p_1) \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 - F_1(t) & F_1(t) \\ F_0(t) & 1 - F_0(t) \end{pmatrix}$$

https://en.wikipedia.org/wiki/Pareto_distribution

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



$$P(T_1 \le t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \ge \tau_1$$

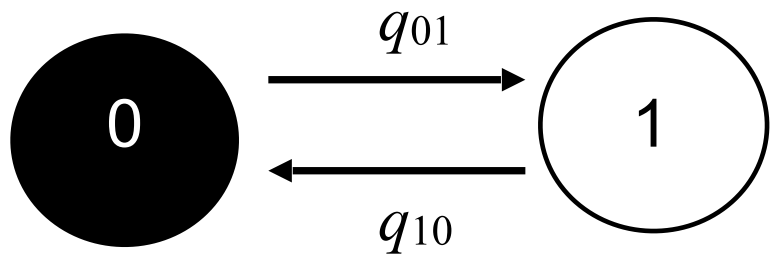
$$P(T_0 \le t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \ge \tau_0$$

$$\vec{p} = (p_0, p_1) \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 - F_1(t) & F_1(t) \\ F_0(t) & 1 - F_0(t) \end{pmatrix}$$

https://en.wikipedia.org/wiki/Pareto_distribution

Semi-Markov process - very simple example

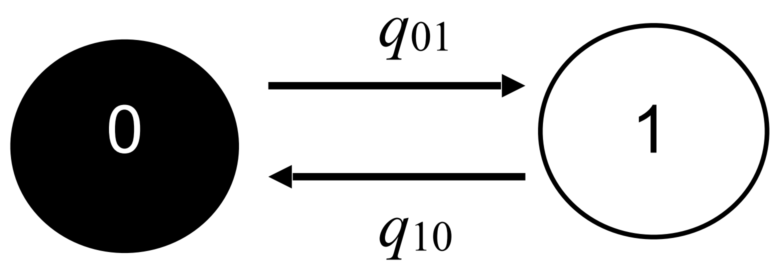
2-state system with Pareto distribution (power law):



https://en.wikipedia.org/wiki/Markov_renewal_process

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



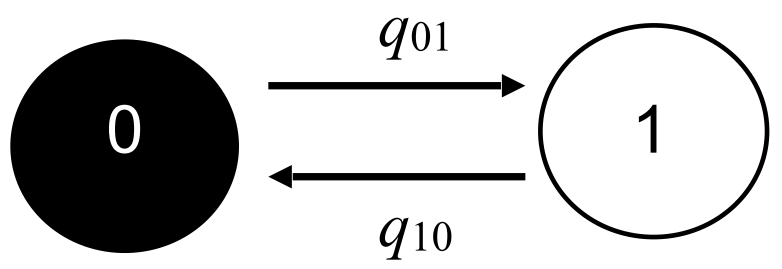
$$P(T_1 \leq t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \geq \tau_1$$

$$P(T_0 \leq t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \geq \tau_0$$

https://en.wikipedia.org/wiki/Markov_renewal_process

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



$$P(T_1 \leq t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \geq \tau_1$$

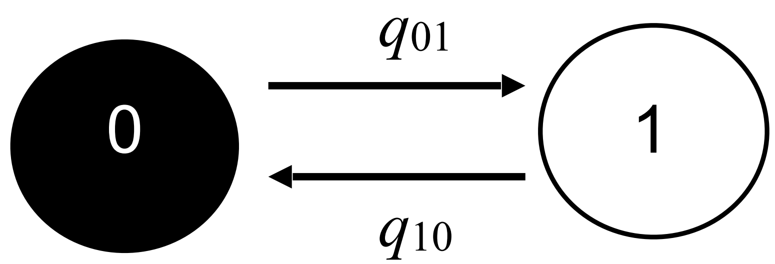
$$P(T_0 \leq t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \geq \tau_0$$

$$p_{01}(t, t+h) = P(X_{t+h} = 1 \,|\, X_t = 0)$$

https://en.wikipedia.org/wiki/Markov_renewal_process

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



$$P(T_1 \le t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \ge \tau_1$$

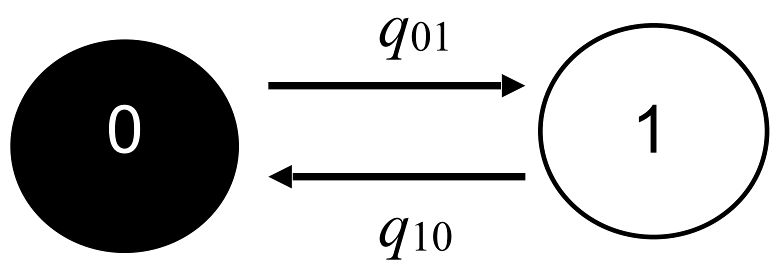
$$P(T_0 \le t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \ge \tau_0$$

$$p_{01}(t, t+h) = P(X_{t+h} = 1 \,|\, X_t = 0) = P(T_1 \le t+h \,|\, T_1 > t) = \frac{P(T_1 \in (t, t+h\rangle)}{P(T_1 > t)}$$

https://en.wikipedia.org/wiki/Markov_renewal_process

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



$$P(T_1 \leq t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \geq \tau_1$$

$$P(T_0 \leq t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \geq \tau_0$$

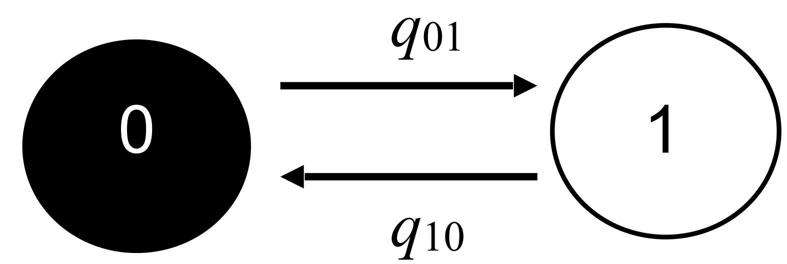
$$p_{01}(t, t+h) = P(X_{t+h} = 1 \,|\, X_t = 0) = P(T_1 \leq t + h \,|\, T_1 > t) = \frac{P(T_1 \in (t, t+h\rangle)}{P(T_1 > t)}$$

$$= \frac{1 - \left(\frac{t+h}{\tau_1}\right)^{-\alpha} - \left[1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}\right]}{\left(\frac{t}{\tau_1}\right)^{-\alpha}} = 1 - \left(\frac{t+h}{t}\right)^{-\alpha}$$

https://en.wikipedia.org/wiki/Markov_renewal_process

Semi-Markov process - very simple example

2-state system with Pareto distribution
(power law):



$$P(T_1 \leq t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \geq \tau_1$$

$$P(T_0 \leq t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \geq \tau_0$$

$$p_{01}(t, t+h) = P(X_{t+h} = 1 \,|\, X_t = 0) = P(T_1 \leq t+h \,|\, T_1 > t) = \frac{P(T_1 \in (t, t+h\rangle)}{P(T_1 > t)}$$

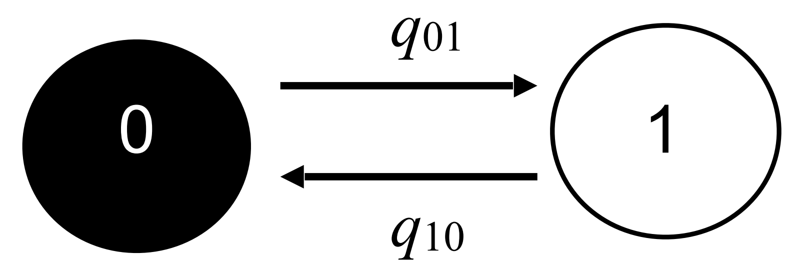
$$= \frac{1 - \left(\frac{t+h}{\tau_1}\right)^{-\alpha} - \left[1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}\right]}{\left(\frac{t}{\tau_1}\right)^{-\alpha}} = 1 - \left(\frac{t+h}{t}\right)^{-\alpha}$$

$$\left(\frac{t+h}{\tau_1}\right)^{-\alpha} = 1 - \frac{\alpha}{t}h + o(h), \quad h \to 0^+$$

https://en.wikipedia.org/wiki/Markov_renewal_process

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



$$P(T_1 \leq t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \geq \tau_1$$

$$P(T_0 \leq t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \geq \tau_0$$

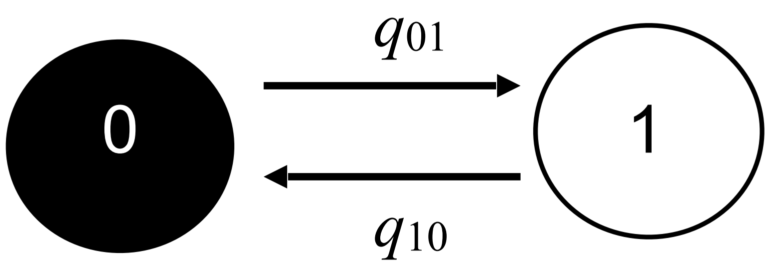
$$p_{01}(t, t+h) = P(X_{t+h} = 1 \,|\, X_t = 0) = P(T_1 \leq t+h \,|\, T_1 > t) = \frac{P(T_1 \in (t, t+h\rangle)}{P(T_1 > t)}$$

$$= \frac{1 - \left(\frac{t+h}{\tau_1}\right)^{-\alpha} - \left[1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}\right]}{\left(\frac{t}{\tau_1}\right)^{-\alpha}} = 1 - \left(\frac{t+h}{t}\right)^{-\alpha} = \frac{\alpha}{t}h + o(h), \quad h \to 0^+$$

https://en.wikipedia.org/wiki/Markov_renewal_process

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



$$P(T_1 \le t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \ge \tau_1$$

$$P(T_0 \le t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \ge \tau_0$$

$$p_{01}(t, t+h) = \frac{\alpha}{t} h + o(h), \quad h \to 0^+$$

$$p_{10}(t, t+h) = \frac{\beta}{t} h + o(h), \quad h \to 0^+$$

Kolmogorov dif. eq.:

$$p_0'(t) = -\frac{\alpha}{t} p_0(t) + \frac{\beta}{t} p_1(t)$$

$$p_1'(t) = \frac{\alpha}{t} p_0(t) - \frac{\beta}{t} p_1(t)$$

$$p_0(0) = 1, \quad p_1(0) = 0$$

Stationary distribution: $\lim_{t \to \infty} \vec{p}(t) = \vec{\pi}$

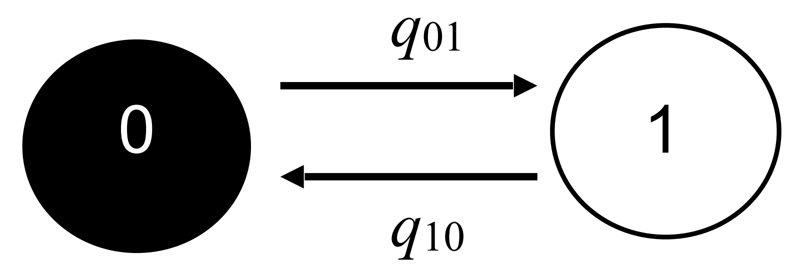
$$0 = \vec{\pi} \cdot Q$$

$$\pi_0 = \frac{\beta}{\alpha + \beta}, \quad \pi_1 = \frac{\alpha}{\alpha + \beta}$$

https://en.wikipedia.org/wiki/Pareto_distribution

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



$$P(T_1 \le t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \ge \tau_1$$

$$P(T_0 \le t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \ge \tau_0$$

$$p_{01}(t, t+h) = \frac{\alpha}{t} h + o(h), \quad h \to 0^+$$

$$q_{01}(t) = \frac{\alpha}{t}$$

$$p_{10}(t, t+h) = \frac{\beta}{t} h + o(h), \quad h \to 0^+$$

$$q_{10}(t) = \frac{\beta}{t}$$

Kolmogorov dif. eq.:

$$p_0'(t) = -\frac{\alpha}{t} p_0(t) + \frac{\beta}{t} p_1(t)$$

$$p_1'(t) = \frac{\alpha}{t} p_0(t) - \frac{\beta}{t} p_1(t)$$

$$p_0(0) = 1, \quad p_1(0) = 0$$

Stationary distribution: $\lim_{t \to \infty} \vec{p}(t) = \vec{\pi}$

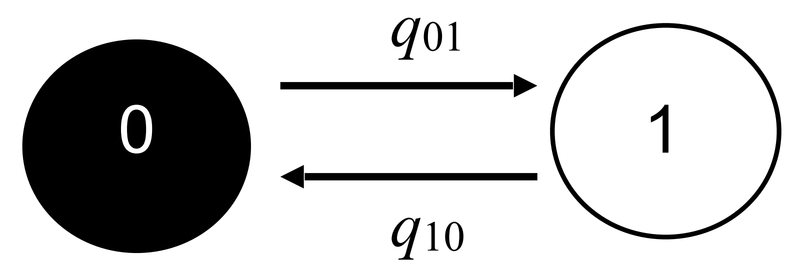
$$0 = \vec{\pi} \cdot Q$$

$$\pi_0 = \frac{\beta}{\alpha + \beta}, \quad \pi_1 = \frac{\alpha}{\alpha + \beta}$$

https://en.wikipedia.org/wiki/Pareto_distribution

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):



$$P(T_1 \leq t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \geq \tau_1$$

$$P(T_0 \leq t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \geq \tau_0$$

$$p_{01}(t, t+h) = \frac{\alpha}{t} h + o(h), \quad h \to 0^+$$

$$p_{10}(t, t+h) = \frac{\beta}{t} h + o(h), \quad h \to 0^+$$

$$q_{00}(t) = -\frac{\alpha}{t} \qquad q_{01}(t) = \frac{\alpha}{t}$$

$$q_{10}(t) = \frac{\beta}{t} \qquad q_{11}(t) = -\frac{\beta}{t}$$

Kolmogorov dif. eq.:

$$p_0'(t) = -\frac{\alpha}{t} p_0(t) + \frac{\beta}{t} p_1(t)$$

$$p_1'(t) = \frac{\alpha}{t} p_0(t) - \frac{\beta}{t} p_1(t)$$

$$p_0(0) = 1, \quad p_1(0) = 0$$

Stationary distribution: $\lim_{t \to \infty} \vec{p}(t) = \vec{\pi}$

$$0 = \vec{\pi} \cdot Q$$

$$\pi_0 = \frac{\beta}{\alpha + \beta}, \quad \pi_1 = \frac{\alpha}{\alpha + \beta}$$

https://en.wikipedia.org/wiki/Pareto_distribution

Semi-Markov process - very simple example

2-state system with Pareto distribution (power law):

$$0 \underset{q_{10}}{\overset{q_{01}}{\rightleftarrows}} 1$$

$$P(T_1 \leq t) = F_1(t) = 1 - \left(\frac{t}{\tau_1}\right)^{-\alpha}, \quad t \geq \tau_1$$

$$P(T_0 \leq t) = F_0(t) = 1 - \left(\frac{t}{\tau_0}\right)^{-\beta}, \quad t \geq \tau_0$$

$$p_{01}(t, t+h) = \frac{\alpha}{t} h + o(h), \quad h \to 0^+$$

$$p_{10}(t, t+h) = \frac{\beta}{t} h + o(h), \quad h \to 0^+$$

$$q_{00}(t) = -\frac{\alpha}{t} \qquad q_{01}(t) = \frac{\alpha}{t}$$

$$q_{10}(t) = \frac{\beta}{t} \qquad q_{11}(t) = -\frac{\beta}{t}$$

$$Q = \begin{pmatrix} -\alpha/t & \alpha/t \\ \beta/t & -\beta/t \end{pmatrix}$$

Kolmogorov dif. eq.:

$$p_0'(t) = -\frac{\alpha}{t} p_0(t) + \frac{\beta}{t} p_1(t)$$

$$p_1'(t) = \frac{\alpha}{t} p_0(t) - \frac{\beta}{t} p_1(t)$$

$$p_0(0) = 1, \quad p_1(0) = 0$$

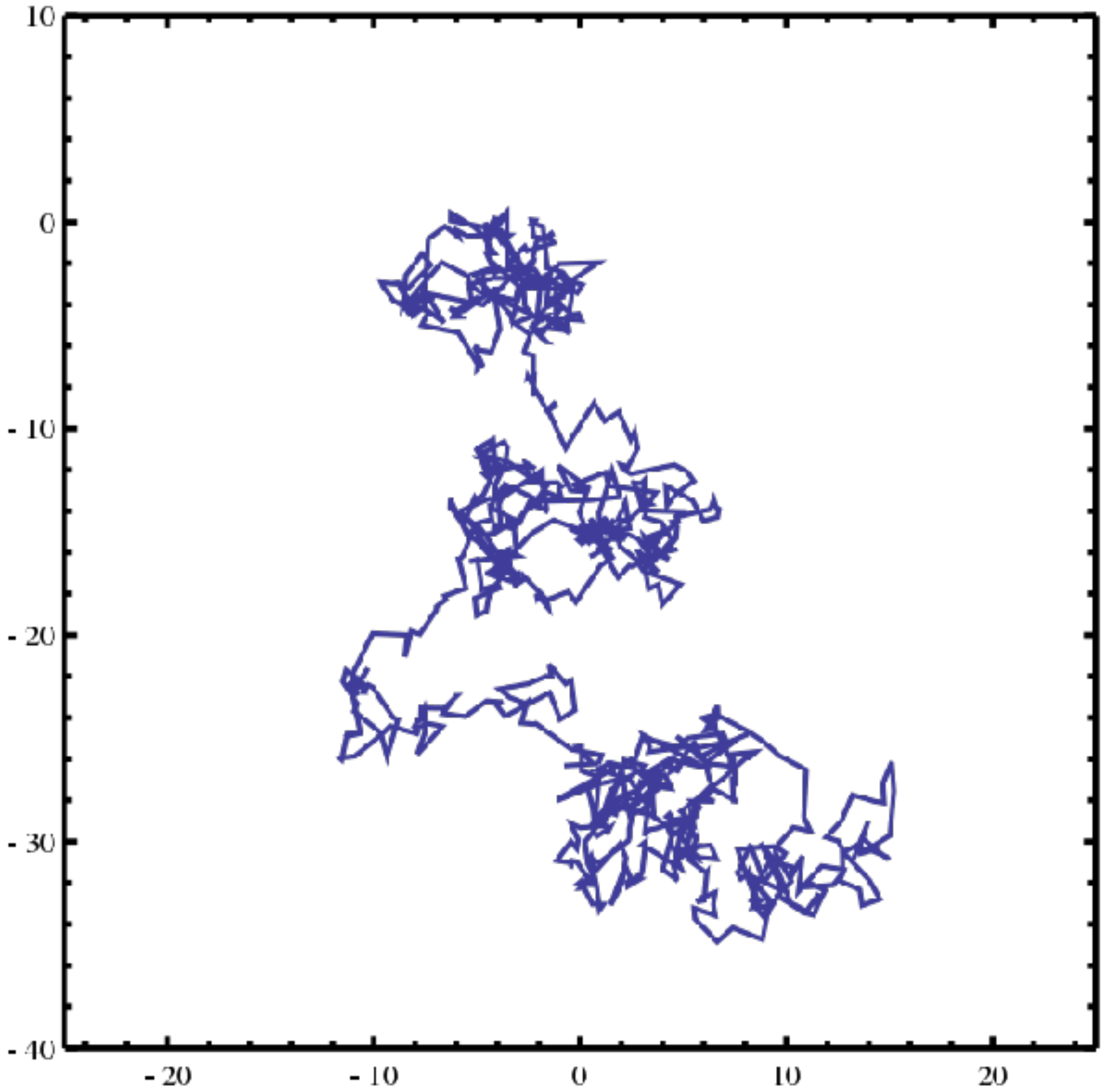
Stationary distribution: $\lim_{t \to \infty} \vec{p}(t) = \vec{\pi}$

$$0 = \vec{\pi} \cdot Q$$

$$\pi_0 = \frac{\beta}{\alpha + \beta}, \quad \pi_1 = \frac{\alpha}{\alpha + \beta}$$

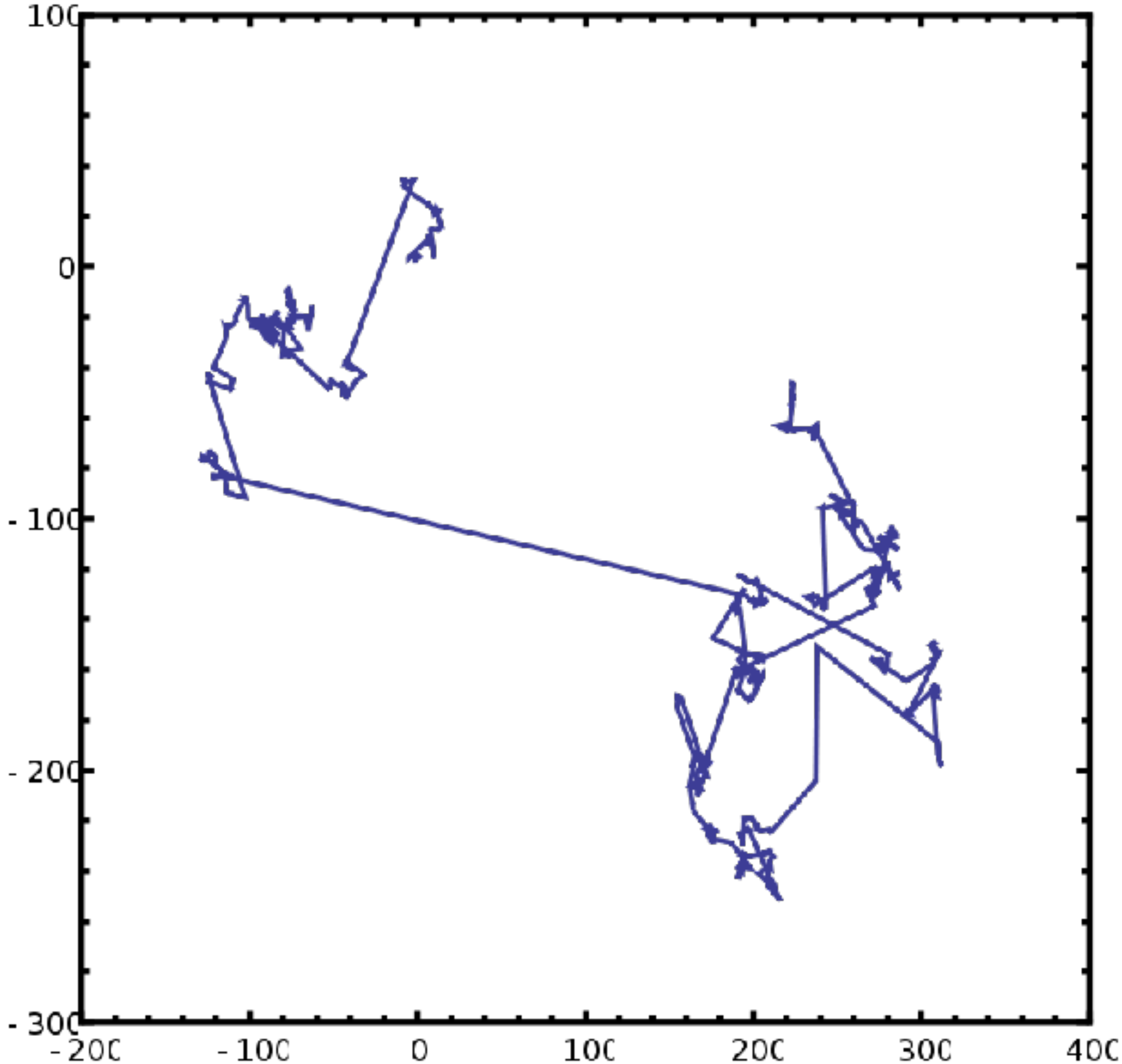
https://en.wikipedia.org/wiki/Pareto_distribution

Markov process - random walk (Brownian motion)



https://en.wikipedia.org/wiki/Brownian_motion

Markov process - random walk (Lévi flight)



https://en.wikipedia.org/wiki/Lévy_flight

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

Thank you for your attention :-)

Gejza Dohnal

Faculty of Mechanical Engineering
Centre of Advanced Aerospace Technologies,
Project CZ.02.1.01/0.0/16_019/0000826

FAKULTA STROJNÍ ČVUT V PRAZE

CAAT